Geometry and combinatorics of Coxeter groups

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Coxeter groups play a fundamental role in several areas of mathematics: they occur as Weyl groups in Lie theory, Kazhdan-Lusztig theory, for Cluster algebras or in algebraic geometry; they are the discrete reflection groups acting on spaces of constant curvature in geometry and they are fundamental to define buildings in geometric group theory. Properties of these groups are often key to a deep understanding of the main relevant structures for these areas.

We will start by covering the basics of Coxeter group theory: exchange/deletion conditions, Matsumoto theorem, geometric representations and root systems. We will apply the theory to show that any discrete group generated by reflections in spherical, Euclidean or hyperbolic geometry is a Coxeter group.

Then we will be discussing the interplay between root systems, the weak order, the Bruhat order and the Cayley graph with its word-metric. We will end this part by showing that Coxeter groups are automatic (Brink-Howlett theorem).

The final part of this class will be dedicated to current research developments. In particular, we will be focusing our attention on Garside shadows, Shi arrangements and their relationship with the (still open) word problem in Artin-Tits (braid) groups and the bi-automaticity of Coxeter groups.

Bibliography (selected)

Books

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