**Geometry and combinatorics of Coxeter groups**

Problems, list 1

Primavera 2024, Università di Bologna

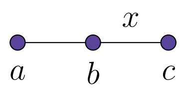
Christophe Hohlweg
hohlweg.christophe@uqam.ca

**Exercise 1** (Length, distance and sign)**.** Let $(W,S)$ be a Coxeter system with length function $\ell : W \to \mathbb{N}$.

a) Show that for any $u, v \in W$, $|\ell(u) - \ell(v)| \leq \ell(uv) \leq \ell(u) + \ell(v)$.
Deduce that $d : W \times W \to \mathbb{N}$, defined by $d(u,v) = \ell(u^{-1}v)$, is a *metric* on $W$.

b) Show that there is a unique group epimorphism $\epsilon : W \to \{\pm 1\}$ such that $\epsilon(s) = -1$ for all $s \in S$. (This morphism is called *the sign of* $(W,S)$).
Deduce that $\ell(ws) = \ell(w) \pm 1$ for all $s \in S$ and $w \in W$.

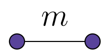
**Exercise 2** (from Björner-Brenti)**.** Let $(W,S)$ be a Coxeter system with $S = \{a,b,c\}$. The Coxeter graph of $(W,S)$ is

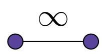



a) Assume that $bcbcacababcacbcabacbabacbc = e$. Find $x$.

b) What is the Coxeter matrix of $W$ ?

**Exercise 3** (Groups generated by two involutions are dihedral groups)**.** Let $G$ be a group generated by a set of two distinct involutions $A = \{a,b\}$.

a) Assume that $G$ is finite, show that the cardinality of $G$ is $|G| = 2m$ for some integer $m \geq 2$. Show that $(G,A)$ is a Coxeter system of type   .

b) If $G$ is infinite, show that $(G,A)$ is a Coxeter system of type   .

**Exercise 4** (Reflections)**.** Let $(V,B)$ be a linear Euclidean space, i.e., $V$ is a finite dimensional real vector space and $B : V \times V \to \mathbb{R}$ is a scalar product. An orthogonal reflection is an isometry $s \in O(V)$ such that the set of fixed points of $s$ is a hyperplane $H$ in $V$. In this case, we say that $s$ is *is the reflection through* $H$.

Let $s \in O(V)$ and $H$ be a hyperplane in $V$. Show that $s$ is the reflection through $H$ if and only for all $v \in V$ we have

$$s(v) = v - 2\frac{B(\alpha, v)}{B(\alpha, \alpha)}\alpha,$$

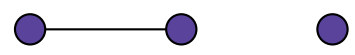
where $\alpha \in V \setminus \{0\}$ is a normal vector to $H$: $\alpha^{\perp} = H$.

**Exercise 5** (Isomorphism of groups vs isomorphism of Coxeter systems)**.** The aim of this exercise is to give an example of two non-isomorphic Coxeter systems $(W_1, S_1)$ and $(W_2, S_2)$ (that is, they are not of the same type) but such that $W_1$ and $W_2$ are isomorphic as abstract groups.

a) Show that the dihedral group $\mathcal{D}_6 \simeq \mathcal{D}_3 \times \mathbb{Z}_2$ (consider the subgroup of $\mathcal{D}_6$ generated by $\sigma = (st)^3$).

b) Consider the dihedral group $\mathcal{D}_3$ in $O(\mathbb{R}^3)$ generated by the reflections $s_1$ and $s_2$ with associated hyperplanes $H_i = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_i = x_{i+1}\}$ (i=1,2).

   Let $H_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ and $s_3$ the associated reflection. Show that $(1, 1, 1)^\perp = H_3$. Deduce that $s_i s_3 = s_3 s_i$ for $i = 1, 2$.

   Now, let $W$ be the reflection group generated by $S = \{s_1, s_2, s_3\}$. Prove that $W \simeq \mathcal{D}_3 \times \mathbb{Z}_2$ and that $(W, S)$ is a Coxeter system of type  .

c) Conclude that there are two non-isomorphic Coxeter systems $(W_1, S_1)$ and $(W_2, S_2)$ such that $W_1$ and $W_2$ are isomorphic as abstract groups