Geometry and combinatorics of Coxeter groups

Problemi, lista 2

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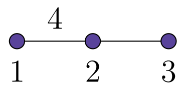
Exercise 1. Let (W, S) be a Coxeter system of type B_3:



We consider (W, S) with its classical geometric representation

a) Find Φ^+ by placing the missing (projective) roots in Figure 1.

b) Deduce the graph of the action of W on $T \times \{\pm 1\}$.

c) Let $w = 2123212132$.

  i) Compute the inversion set of $T(w)$. Is the word 2123212132 reduced for w?

  ii) Compute the set $\mathrm{Red}(w)$ of reduced words for w.

d) Find the types of all reflection subgroups of (W, S). For each type give a set of canonical generators (justify your answer).

Exercise 2 (Strong exchange condition)**.** Let (W, S) be a Coxeter system. Let $\in W$ and $w = s_1 \dots s_k$ a word (not necessarily reduced) for w. Let $t \in T(w)$, show there is $1 \leq i \leq k$ such that $w = ts_1 \dots \hat{s}_i \dots s_k$.

Exercise 3 (Palindromic expression of reflections)**.** Let (W, S) be a Coxeter system. Let $t \in T$ and $t = s_1 \dots s_k$ be a reduced word for t. Show that $k = 2p + 1$ for $p \in \mathbb{N}$ and that $w = s_1 \dots s_{p-1} s_p s_{p-1} \dots s_1$ is a (palindromic) reduced word for t.

Exercise 4. Let (W, S) be a *finite* Coxeter system; denote $w_\circ$ its longest element. Let $w \in W$.

a) Denote $I = D_L(w)$. Write $J = S \setminus I$, then W_I is a finite standard parabolic subgroup of W with longest element $w_{\circ,J} \in W_J$. Show that $w \wedge_R w_{\circ,J} = e$ and $w \vee_R w_{\circ,J} = w_\circ$.

b) Show that $T(ww_\circ) = T \setminus T(w)$ and that $w \wedge_R ww_\circ = e$ and $w \vee_R ww_\circ = w_\circ$.

c) Show that the set $\{x \in W \mid x \wedge_R w \text{ and } x \vee_R w = w_\circ\}$ is an interval in $(W, \leq_R)$.

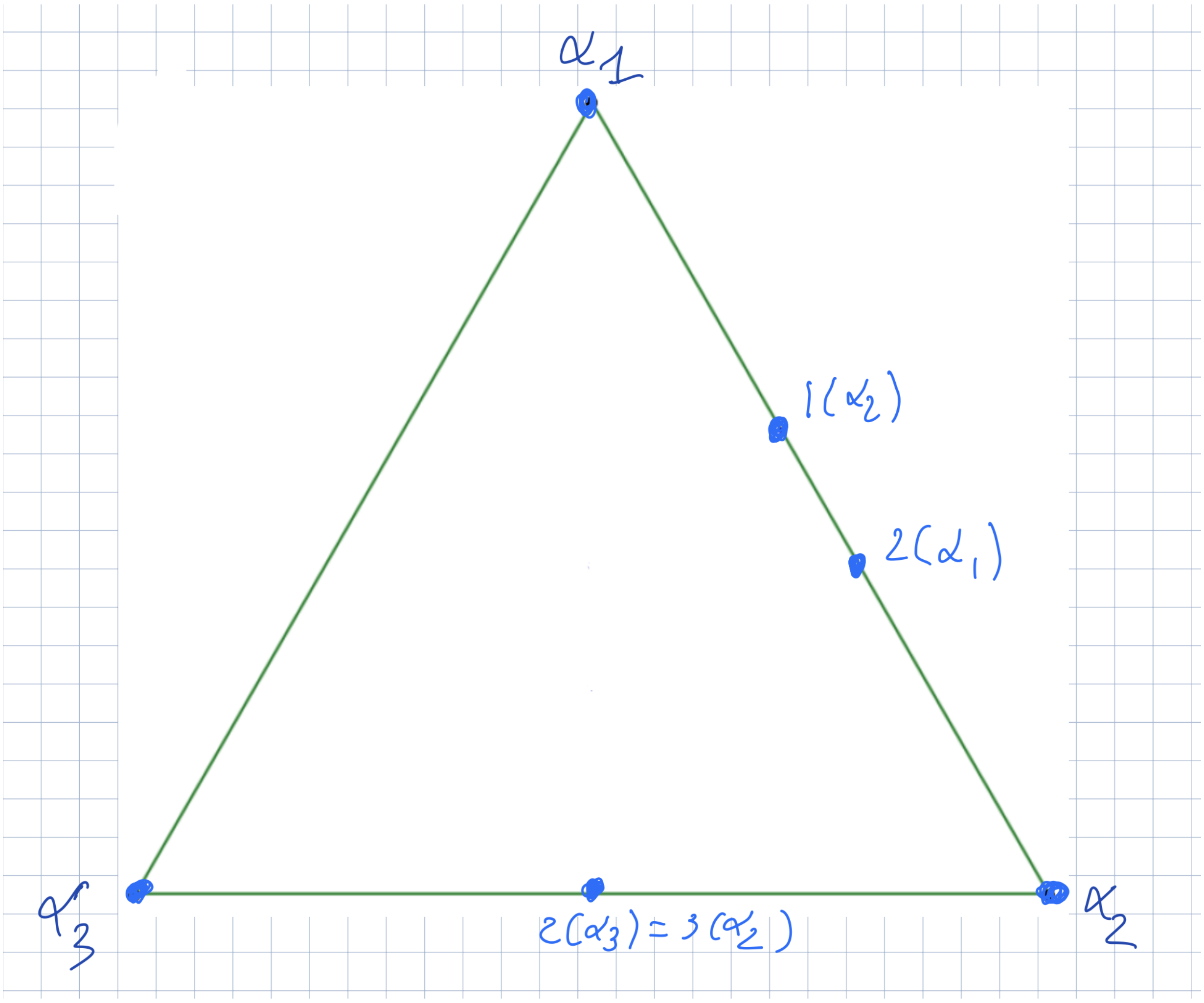


Figure 1: $\mathbb{P}\Phi$ for (W, S) of type B_3 for Exercise 1 - to be completed