Geometry and combinatorics of Coxeter groups

Problemo finale

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Let (W, S) be a Coxeter system. Let (Δ, ϕ) be a based root system in a quadratic space (V, B). For $x, y \in W$, let $\Phi(x, y) = \{\alpha \in \Phi^+ \mid H_\alpha$ separates x and y$\}$.

a) For $x, y \in W$, show that:

   i) $\Phi(xy) = \Phi(x) + x(\Phi(y))$, where $A + B$ is the symmetric difference of A and B.

   ii) $\Phi(x, y) = \Phi(x) + \Phi(y)$.

   iii) $d(x, y) := \ell(x^{-1}y) = |\Phi(x, y)|$.

b) Let $w \in W$ and consider the interval $[e, w]_R$ in $(W, \leq_R)$. Set:

$$d_w := \max\{d(x, y) \mid x, y \in [e, w]_R\}.$$

   Show that:

   i) $d_w = \ell(w)$;

   ii) for $x, y \in [e, w]_R$ we have $d(x, y) = d_w$ if and only if $\Phi(w) = \Phi(x) \sqcup \Phi(y)$. In this case, what is the join $x \vee_R y$?