Permutahedra and Generalized Associahedra

Christophe Hohlweg,

Université du Québec à Montréal

Joint work with Carsten Lange (TU Berlin) and Hugh Thomas (U of New Brunswick)

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How to construct a generalized associahedron

from a permutahedron in 20 minutes.

Framework

(W, S) finite Coxeter system.

Example. $W = S_n$ is generated by $\tau_i = (i\, i+1)$, $1 \leq i < n$.

- **Length of $w \in W$:** $\ell(e) = 0$ and for all $w \neq e$,

$$\ell(w) = \min\{k \geq 1 \,|\, w = s_1 \ldots s_k,\ s_i \in S\}.$$

  Example. $\ell(w) = |\{(i < j) \,|\, w(i) > w(j)\}|$.

- **Reduced expressions for $w \in W$:** $w = s_1 \ldots s_k$ with $k = \ell(w)$.
  Example. $w = \tau_1 = \tau_3 \tau_1 \tau_3$. The second expression is not reduced!

- **Longest element:** w_0 unique element of maximal length.
  Example. $n = 4$, $w_0 = \tau_1\tau_2\tau_3\tau_2\tau_1\tau_2 = \tau_3\tau_1\tau_2\tau_3\tau_1\tau_2$.

Geometric representation of W

$(V, \langle \cdot, \cdot \rangle)$ $\mathbb{R}$-euclidean vector space of dimension $p = |S|$.

- **Simple system:** $\Delta = \{\alpha_s \,|\, s \in S\}$ is basis of V.
  The angle between α_s and α_t is $\pi - \pi/(\text{order of } (st))$.

  Example. $\mathbb{R}^n$ with its canonical basis $\{e_1, e_2, \ldots, e_n\}$.
  Take V spanned by $\Delta = \{\alpha_{\tau_i} = e_{i+1} - e_i \,|\, 1 \leq i < n\}$.
- W **acts by reflection on** V**:**

  $s \in S$ is the (unique) reflection mapping α_s to $-\alpha_s$.
- **Fundamental weights:**

  $\Delta^* = \{v_s \,|\, s \in S\}$ such that $\langle v_s, \alpha_t \rangle = \delta_{s,t}$.

  Example. $v_{\tau_i} = \frac{i-n}{n} \sum_{k=1}^{i} e_k + \frac{i}{n} \sum_{k=i+1}^{n} e_k$.

The Permutahedron $\mathsf{Perm}(W)$

- Set $M(e) = \sum_{s \in S} v_s$ and $M(w) = w\big(M(e)\big)$.

  Example. For $w \in S_n$, $M(w) = (w^{-1}(1), \ldots, w^{-1}(n)) \in \mathbb{R}^n$ (modulo a translation).

- $\mathsf{Perm}(W)$ is the convex hull of $\{M(w) \,|\, w \in W\}$.

- **H-representation:** For $x \in W$ and $s \in S$, we set

$$\mathscr{H}_{(x,s)} = \{v \in V \,|\, \langle v, x(v_s)\rangle \leq \langle M(e), v_s\rangle\}.$$
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  We have

$$\mathsf{Perm}(W) = \bigcap_{s \in S} \bigcap_{x \in W} \mathscr{H}_{(x,s)} \quad \text{and} \quad \{M(w)\} = \bigcap_{s \in S} H_{(w,s)}.$$

- **Coxeter fan:** the normal fan of $\mathsf{Perm}(W)$.

c-sortable elements

Write $S = \{s_1, s_2, \dots, s_p\}$ with $|S| = p$.

- **Coxeter element:** $c = s_1 s_2 \dots s_p$.
  Example. $n = 4$, $c = \tau_1 \tau_3 \tau_2$.
- $c_{(I)}$ subword of c with letters in $I \subseteq S$.
  Example. $n = 4$, $c_{(\{\tau_2, \tau_3\})} = \tau_3 \tau_2$.
- **c-sortable elements:** $w \in W$ is c-sortable $\iff$
  w a reduced expression of the form
  $$c = c_{(K_1)} c_{(K_2)} \dots c_{(K_l)}, \quad K_1 \supseteq K_2 \supseteq \dots \supseteq K_l \neq \emptyset.$$
  This expression is unique and called the *c-factorization of w*. The word in S corresponding to the c-factorization is called the *c-word* of w.

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  $$e, \tau_1, \tau_2, \tau_3, \tau_1\tau_3, \tau_3\tau_2, \tau_1\tau_2,$$
  $$\tau_3\tau_2|\tau_3, \tau_1\tau_2|\tau_1, c, c|\tau_1, c|\tau_3, c|\tau_1\tau_3, w_0 = c|c.$$

Almost positive roots and Last roots

Root system: $\Phi = W(\Delta)$, **Positive root system:** $\Phi^+ = \Phi \cap \mathbb{R}^+[\Delta]$.

- **Almost positive roots:** $\Phi_{\geq -1} = -\Delta \cup \Phi^+$.
  Example. $n = 3$, $\Phi^+ = \{e_2 - e_1, e_3 - e_2, e_3 - e_1\}$ and
  $\Phi_{\geq -1} = \{e_1 - e_2, e_2 - e_3, e_2 - e_1, e_3 - e_2, e_3 - e_1\}$.

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- **Last root corresponding to** $s \in S$ **of a** c**-sortable element** w**:**

$$lr_s(w) = \begin{cases} u(\alpha_s) & \text{if the } c\text{-word of } w \text{ can be written } usv \\ & \text{and } s \text{ is not a letter in } v \\ -\alpha_s & \text{otherwise} \end{cases}$$

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	e	τ_2	$c = \tau_2\tau_1$	$w_0 = \tau_2\tau_1 \vert \tau_2$	τ_1
τ_1	$e_1 - e_2$	$e_1 - e_2$	$e_3 - e_1$	$e_3 - e_2$	$e_2 - e_1$
τ_2	$e_2 - e_3$	$e_3 - e_2$	$e_3 - e_2$	$e_2 - e_1$	$e_2 - e_3$

Clusters

(S. Fomin, A. Zelevinsky; R. Marsh, M. Reineke and A. Zelevinsky, N. Reading)

- c**-clusters:** $\{lr_s(w) \,|\, s \in S\}$ for w c-sortable element.
- Exchange graph: Vertices are the c-clusters
  Edges: $\{x, x'\}$ if $x' = x \cup \alpha \setminus \beta$ for $\alpha, \beta \in \Phi_{\geq -1}$.
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τ_1	$e_1 - e_2$	$e_1 - e_2$	$e_3 - e_1$	$e_3 - e_2$	$e_2 - e_1$
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Generalized Associahedra

The generalized associahedron $\mathsf{Asso}(W)$ was introduced by S. Fomin and A. Zelevinsky in the context of *cluster algebras* (and realized with F. Chapoton).

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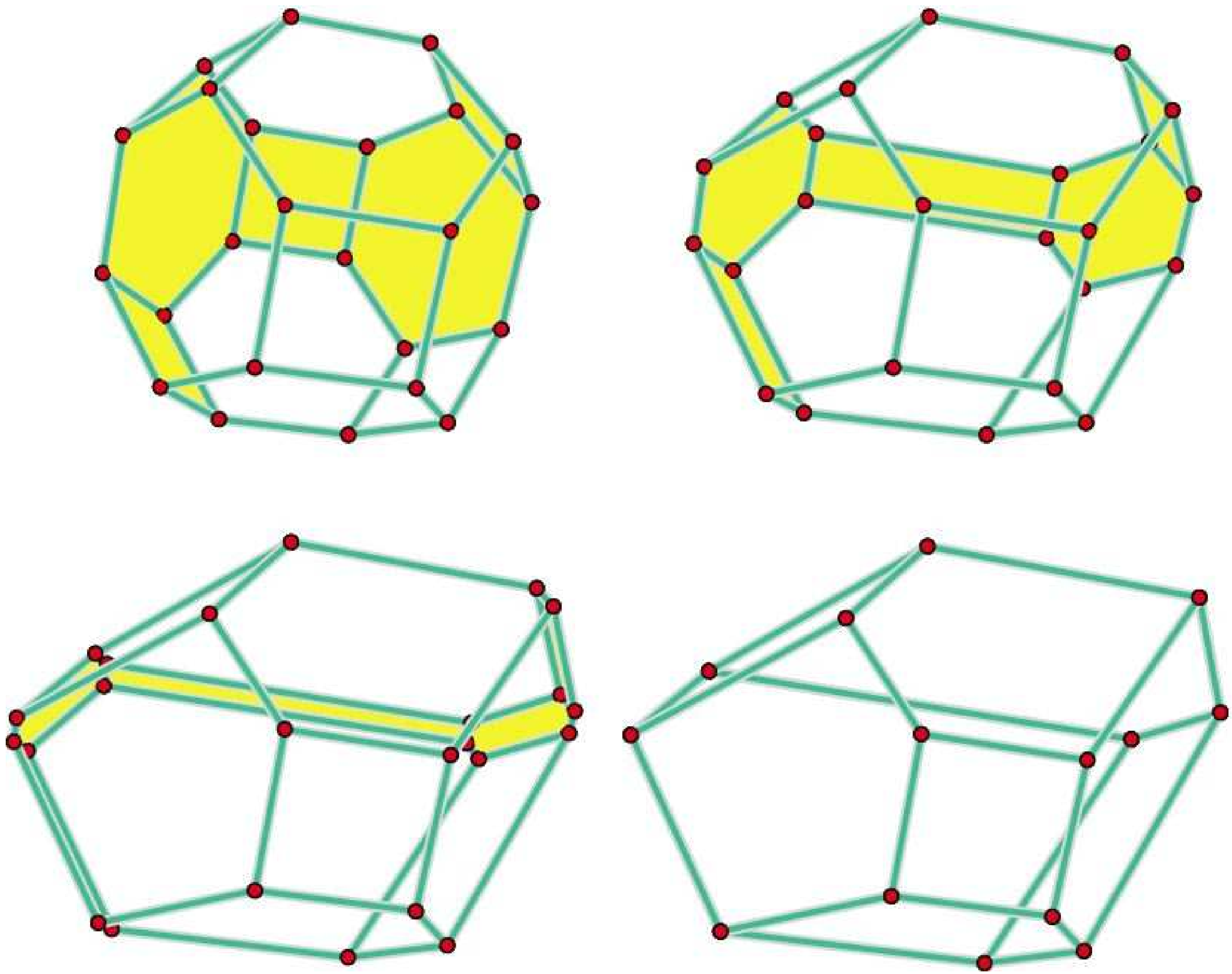
$\text{Asso}(W)$ is a simple $|S|$-dimensional convex polytope.

- **Definition (Combinatorial characterization):** The 1-skeleton of $\text{Asso}(W)$ is isomorph to the exchange graph.
- **Loday's realization:** (Shnider-Sternberg (1995), Loday (2004)) The Associahedron $\text{Asso}(S_n)$ can be obtained by removing facets from the permutahedron
- **Problems:** Loday's realization for W in general?
- **Conjecture:** the c-Cambrian fan is polytopal (N. Reading (2004)).

Loday's realization for S_4



c-singletons

- c**-singleton:** w is a c-singleton if
  - w is c-sortable;
  - ws is c-sortable for all $s \in S$ such that $\ell(ws) > \ell(w)$.
- **Characterization:**

  Theorem. *w is a c-singleton if and only if w is a prefix of the c-factorization of w_0 (up to allowed commutation of simple reflections).*

  Example. $n = 4$, $c = \tau_1\tau_3\tau_2$, the c-factorization of w_0 is $w_0 = \tau_1\tau_3\tau_2|\tau_1\tau_3\tau_2$.

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  $$w_0 \,;\, \tau_1\tau_3\tau_2|\tau_1\tau_3 \,;\, \tau_1\tau_3\tau_2|\tau_1 \,;\, \tau_1\tau_3\tau_2 = c \,;\, \tau_1\tau_3 \,;\, \tau_1 \,;\, e$$

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Example. $n = 4$, $c = \tau_1\tau_3\tau_2$, the c-factorization of w_0 is $w_0 = \tau_1\tau_3\tau_2 | \tau_1\tau_3\tau_2$. The c-singletons are

$$w_0 \,;\, \tau_1\tau_3\tau_2 | \tau_1\tau_3 \,;\, \tau_1\tau_3\tau_2 | \tau_1 \,;\, \tau_1\tau_3\tau_2 = c \,;\, \tau_1\tau_3 \,;\, \tau_1 \,;\, e$$

But also $\tau_1\tau_3\tau_2 | \tau_3$; τ_3 obtained from $w_0 = \tau_3\tau_1\tau_2 | \tau_3\tau_1\tau_2$

Construction of Generalized Associahedra

Remind that

$$\mathsf{Perm}(W) = \bigcap_{s \in S} \bigcap_{x \in W} \mathscr{H}_{(x,s)} \quad \text{and} \quad \{M(w)\} = \bigcap_{s \in S} H_{(w,s)}.$$

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  $\mathscr{H}_{(x,s)}$ is c-admissible if $H_{(x,s)}$ contains a c-singleton.

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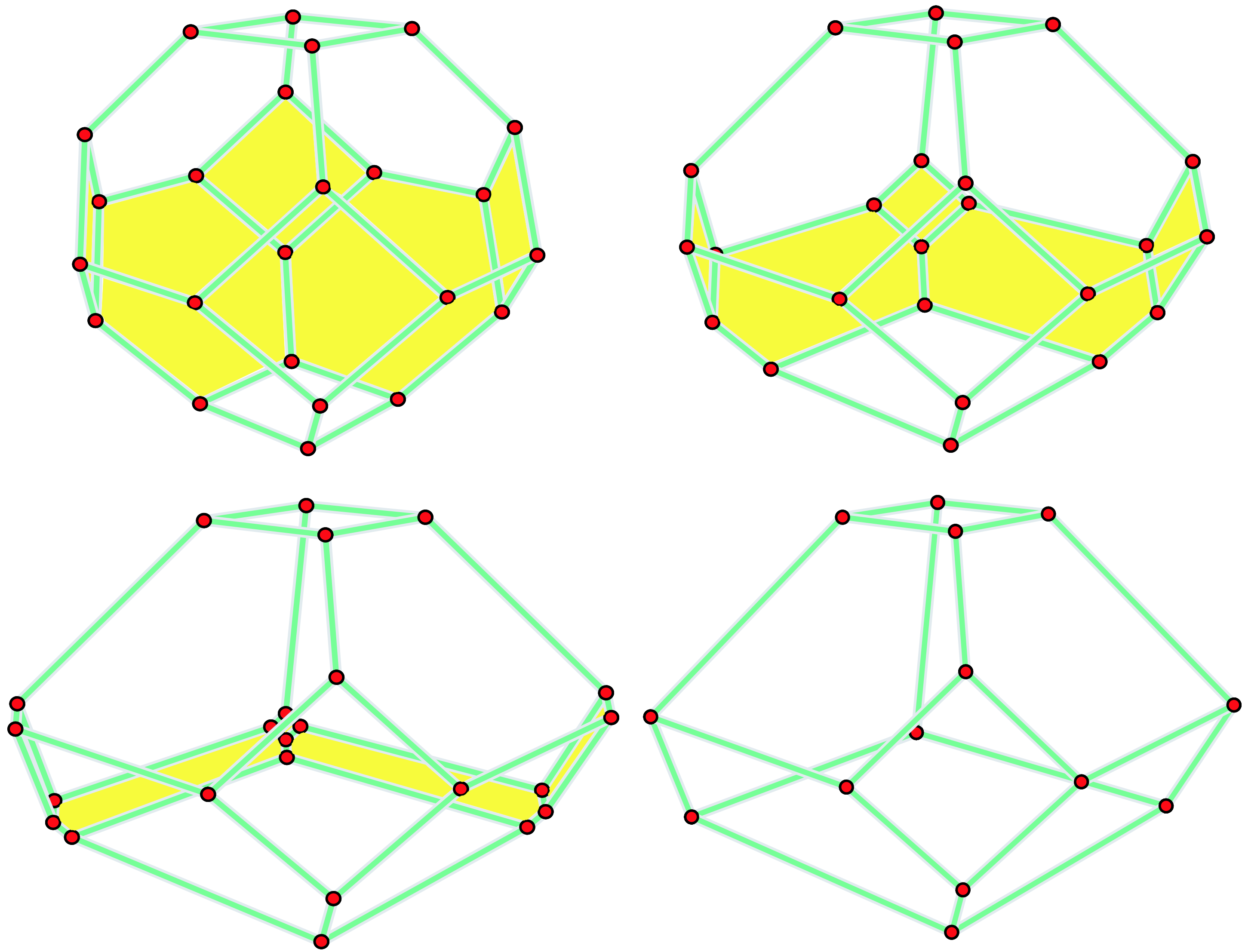
Theorem. *The intersection* $\mathsf{Asso}_c(W)$ *of the c-admissible half-spaces is a generalized associahedron (i.e. a simple convex polytope combinatorially isomorph to* $\mathsf{Asso}(W)$*).*
The normal fan of $\mathsf{Asso}_c(W)$ *is the c-Cambrian fan.*
The map $\mathscr{H}_{(x,s)} \mapsto lr_s(x)$ for x a c-singleton in $H_{(x,s)}$ is a bijection between the c-admissible half-spaces and $\Phi_{\geq -1}$.

Example $W = S_4$ with $c = \tau_1\tau_3\tau_2$



Remarks

- **Classical type A and B** (H-Lange (2005)): We obtain integer coordinates for $\text{Asso}_c(W)$ combinatorially computed from the triangulations.
- **In general:** If $M(w)$ has integer coordinates in a basis (i.e. W is a Weyl group), then $\text{Asso}_c(W)$ too.
- **Isometry classes**: These realizations are not all isometric. A classification of isometry classes is known (Bergeron, H, Lange, Thomas (2007))
- **Conjecture:** the isobarycenter (center of gravity) of the vertices is 0 for $M(e) = \sum_{s\in S} v_s$.
- $|\{c$-sortable elements$\}|$= W-Catalan number (N. Reading, 2006) What is the number of c-singletons (does depend on c)?