The Solomon Descent Algebra

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S_n symmetric group acting on $[1, n] = \{1, \ldots, n\}$

- *Descent set of* $w \in S_n$

$$D(w) = \{i \in [1, n-1] \,|\, w(i) > w(i+1)\}$$

- *Descent representations* (Solomon $\simeq$1968)

$$\mathbb{Q}S_n = \bigoplus_{I \subset [1,n-1]} V_I$$

The dimension of the S_n-module V_I is:

$$\dim V_I = |\{w \in S_n \,|\, D(w) = I\}|\,.$$

S_n symmetric group acting on $[1,n] = \{1,\ldots,n\}$

- *Cycle type representations* (Garsia–Reutenauer, 1989)

$$\mathbb{Q}S_n = \bigoplus_{\lambda \vdash n} E_\lambda \mathbb{Q}S_n$$

The dimension of the S_n-module $E_\lambda \mathbb{Q}S_n$:

$$\dim E_\lambda \mathbb{Q}S_n = \left|\{w \in S_n \text{ having cycle type } \lambda\}\right| .$$

A Gessel–Reutenauer formula (1993)

$$\langle \chi_I, \chi_\lambda \rangle = |\{w \in C(\lambda) \mid D(w) = I\}|$$

where

- χ_I is the character of V_I (*descent character*).
- χ_λ is the character of $E_\lambda \mathbb{Q} S_n$.

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How to generalize this formula to finite Coxeter groups?

Symmetry property

We extend linearly θ to $\mathbb{Q}W$.

Theorem (Blessenohl, H., Schocker).

$$\theta(x)(y) = \theta(y)(x), \quad \forall x, y \in \Sigma W.$$

Proof. Combinatoric of root systems (hard)

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Proof. $\langle \chi_\lambda, \chi_I \rangle = \chi_I(E_\lambda) = \theta(d_I)(E_\lambda) = \theta(E_\lambda)(d_I) = \xi_\lambda(d_I)$

And for conjugacy classes and descent sets?

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- For the *hyperoctahedral group* W_n (type B_n),

  we (joint with P. Baumann) obtain the formula

$$\langle \chi_C, \chi_I \rangle = |\{w \in C \mid D'(w) = I\}|$$

  * the χ_I are the characters of the *generalized descent representations* (Adin, Brenti, Roichman - 2001).
  * the χ_C are the character of certain W_n-modules associated to the decomposition of a '*generalized descent algebra*'.

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- A conjecture (N. Bergeron 90', G. Pfeiffer 00'):

  $\chi_\lambda = \sum_{C \subset C(\lambda)} \chi_C$ such that

  - $\langle \chi_\lambda, \chi_I \rangle = |\{w \in C \mid D(w) = I\}|$;
  - $\chi_C = \mathrm{Ind}_{\mathrm{Fix}(w_C)}^{W}$ of a linear character on $\mathrm{Fix}(w_C)$.

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This is the End