

Theory of Coxeter Groups

~ An INTRODUCTION ~

CIME SCHOOL: Algebraic & geometric combinatoric
Coxeter groups and representation theory.

Christophe HOHLWEG (UQAM, MONTRÉAL)

(visiting Università di Roma "TOR VERGATA" & "SAPIENZA" - winter/spring 2026)

PLAN OF THE TALK

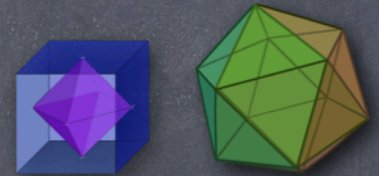
I. INTRODUCTION AND SOME MOTIVATIONS

II. How to work with Coxeter systems

I. INTRODUCTION & MOTIVATIONS

A bit of history

- Classification of regular polygons & polyhedral (cf. Euclid 300BC)
- Study of regular tilings of the plane and the sphere (Byzantine school, High Middle-age, Kepler 1619)



19th – century

- Study of (discrete groups of) isometries (Möbius, Jordan)
- Study of Tilings and regular polytopes (Schläfli)



20th – century: Coxeter groups, Lie Theory and beyond

- Classification of discrete reflection groups on S^n , E^n (Coxeter 1934)
- Classification of root systems (Cartan, Killing, Weyl, Dynkin 1880–1947)
- Discrete reflection groups on H^n are Coxeter groups (Vinberg 1971)



(Images: wikipedia)

Since then: connections to Algebraic geometry, Arithmetic & Geometric group theory, Combinatorics, Representation theory, physics ...

INTUITIVELY: COXETER GROUPS ARE GROUPS GENERATED BY REFLECTIONS ACTING ON LOCALLY FINITE TILINGS.

Coxeter Groups (ABSTRACT DEF.)

Coxeter graph Γ :

- Vertices S

- edges $\overset{s}{\bullet} \overset{m_{st}}{\text{---}} \overset{t}{\bullet}$ with $m_{st} \geq 3$ or $m_{st} = \infty$

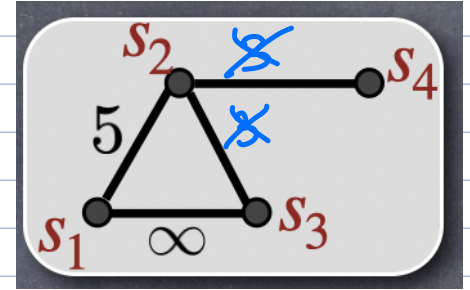
Define $m_{st} = 2$ if no edge $\overset{s}{\bullet} \overset{t}{\bullet}$

COXETER SYSTEM (W, S) of TYPE Γ :

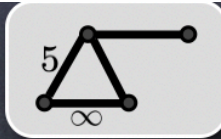
$$W = \langle S \mid s^2 = (st)^{m_{st}} = e, \text{ if } m_{st} \neq \infty \rangle$$

Remark: (i) $m_{ss} = 1 \Leftrightarrow s^2 = e$ and $m_{st} = \infty \Leftrightarrow$ no relation between s, t

(ii) $m_{st} = 2 \Leftrightarrow st = ts \Leftrightarrow$ no edge $\overset{s}{\bullet} \overset{t}{\bullet}$ in Γ



Example 1. (W, S) of type



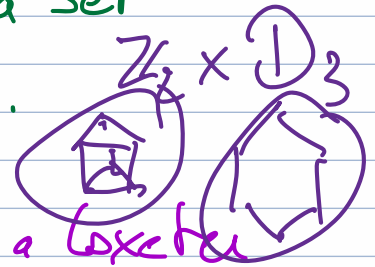
above is:

$$W = \langle s_1, s_2, s_3, s_4 \mid s_i^2 = (s_1 s_2)^5 = (s_2 s_3)^3 = (s_2 s_4)^3 = (s_1 s_4)^2 = (s_3 s_4)^2 = e \rangle$$

PROPOSITION (exercise) Any Coxeter system (W, S) has a sign character:

$$\begin{aligned} \varepsilon: W &\longrightarrow \{\pm 1\} \text{ group epimorphism. In part.: } \underbrace{\text{ord}_W(s) = 2, \forall s \in S.}_{\text{involution: } s^2 = e, s \neq e} \\ s \in S &\longmapsto -1 \end{aligned}$$

DEF (COXETER GROUP) A group W for which there is a set of involution S such that (W, S) is a Coxeter system.



- It is undecidable to determine if a finitely presented group has a Coxeter structure (Adyan - Rabin theorem, Markov property).
- When a group is a Coxeter group, it is open to determine all its Coxeter systems (W, S)

$$(12)(23) = (123)$$

The "tools" develop here depend on the choice of the Coxeter system

Example 2. The symmetric group $\mathcal{S}_n$ generated by the simple transpositions $\tau_i = (i \ i+1)$, $1 \leq i < n$:



$$\mathcal{S}_n = \langle \tau_i \mid \tau_i^2 = (\tau_i \tau_j)^2 = (\tau_i \tau_{i+1})^3 = e, |i - j| > 1 \rangle$$

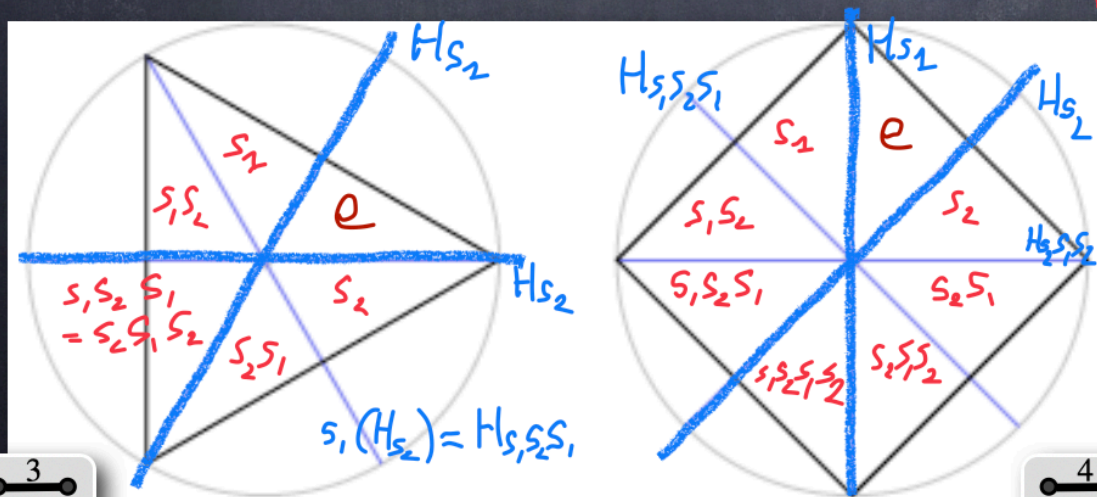
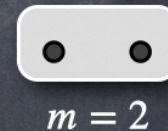
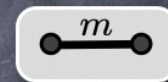
$\mathbb{E}$ Euclidean space of dim. $n \in \mathbb{N}^*$

W discrete group generated by Euclidean reflections

T is the set of reflections in W ; $\mathcal{A} = \{H_t \mid t \in T\}$ Coxeter arrangement

Theorem (Coxeter, 1934). There exists $S \subseteq T$ such that (W, S) is a Coxeter system. Moreover $T = \bigcup_{w \in W} wSw^{-1}$.

Example 3 ($n = 2$): Dihedral groups $\mathcal{D}_m$, the isometry group of a regular m -gon ($m \in \mathbb{N}_{\geq 2}$): $\mathcal{D}_m = \langle s_1, s_2 \mid s_i^2 = \underbrace{(s_1 s_2)^m}_{\text{ROTATION } \frac{2\pi}{m}} = e \rangle$.



1. Choose $C = e(C)$ a connected comp. of $\mathbb{E} \setminus \bigcup_{t \in T} H_t$ (fund. chamber)
2. S = reflections across wall of C
3. $W \leftrightarrow$ chambers (bijection)

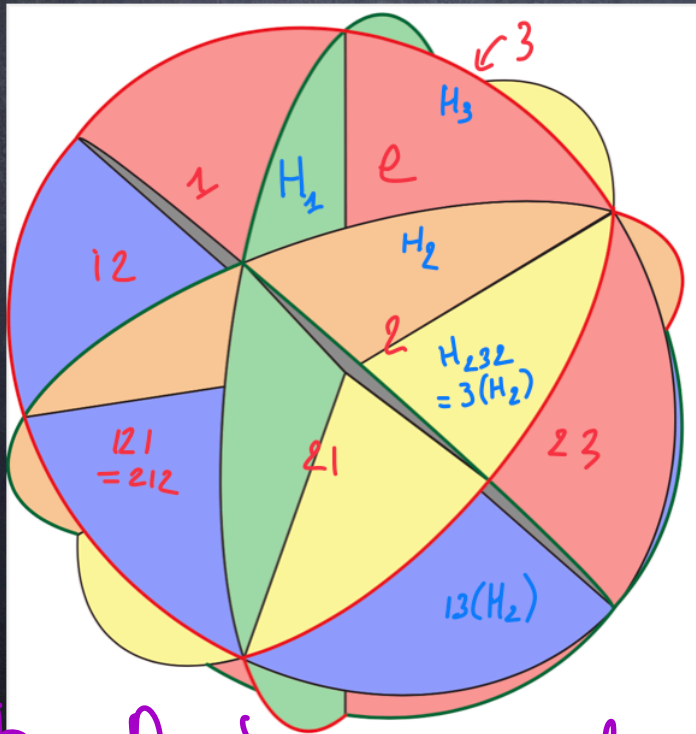
Rk: m = order of the rotation $s_1 s_2$

$\mathbb{E}$ Euclidean space of dim. $n \in \mathbb{N}^*$

W discrete group generated by Euclidean reflections

T is the set of reflections in W ; $\mathcal{A} = \{H_t \mid t \in T\}$ Coxeter arrangement

Theorem (Coxeter, 1934). There exists $S \subseteq T$ such that (W, S) is a Coxeter system. Moreover $T = \bigcup_{w \in W} wSw^{-1}$.



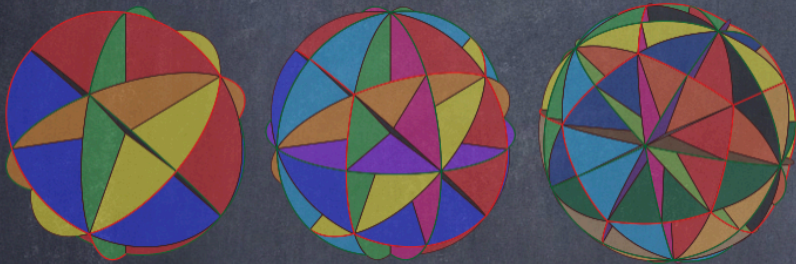
Example $\nexists$ ($n = 3$) $W = \mathcal{S}_4$

1. Choose $C = e(C)$ a connected comp. of $\mathbb{E} \setminus \bigcup_{t \in T} H_t$ (fund. chamber)
2. S = reflections across wall of C
3. $W \leftrightarrow$ chambers (bijection)

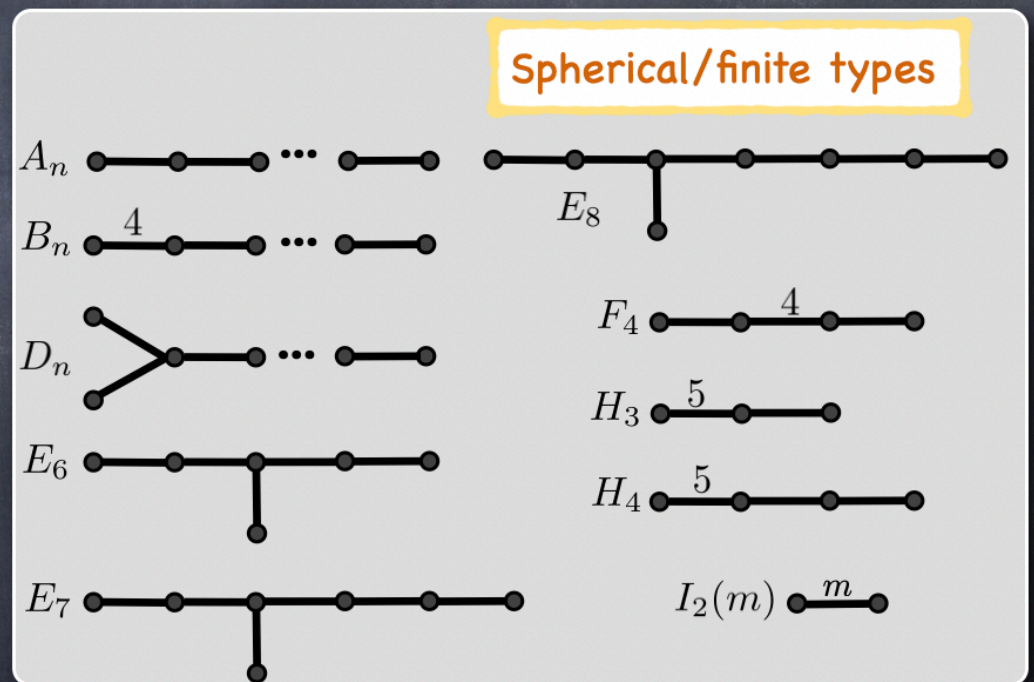
Rk: Again m = order of the rotation $s_1 s_2$

PROPOSITION (EXERCISE) Let W discrete subgroup of $\text{Isom}(\mathbb{E}^n)$.
 Then W is finite $\Leftrightarrow W$ fixes at least one point in $\mathbb{E}^n$

Theorem (Coxeter, 1934): W is finite iff Γ has connected components of spherical types.



Regular tilings of the sphere

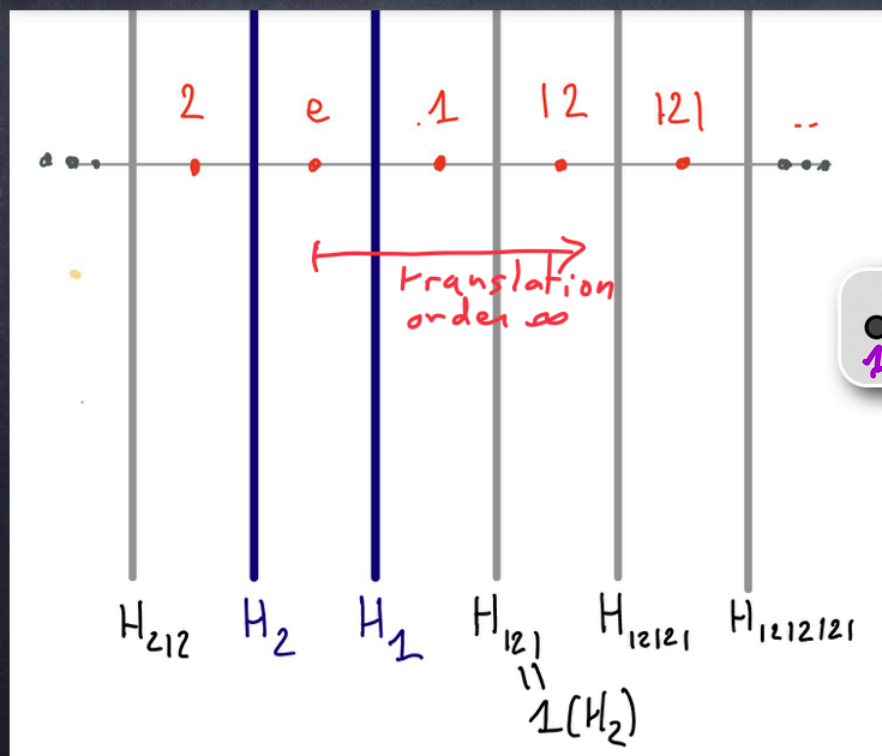


$\mathbb{E}$ Euclidean space of dim. $n \in \mathbb{N}^*$

W discrete group generated by Euclidean reflections

T is the set of reflections in W ; $\mathcal{A} = \{H_t \mid t \in T\}$ Coxeter arrangement

Theorem (Coxeter, 1934). There exists $S \subseteq T$ such that (W, S) is a Coxeter system. Moreover $T = \bigcup_{w \in W} wSw^{-1}$.



Example $\mathfrak{S}(n=2)$ $W = \mathcal{D}_\infty$
the infinite dihedral group

$$\mathcal{D}_\infty = \langle s_1, s_2 \mid s_i^2 = e \rangle$$

1. Choose $C = e(C)$ a connected comp. of $\mathbb{E} \setminus \bigcup_{t \in T} H_t$ (fund. chamber)
2. S = reflections across wall of C
3. $W \leftrightarrow$ chambers (bijection)

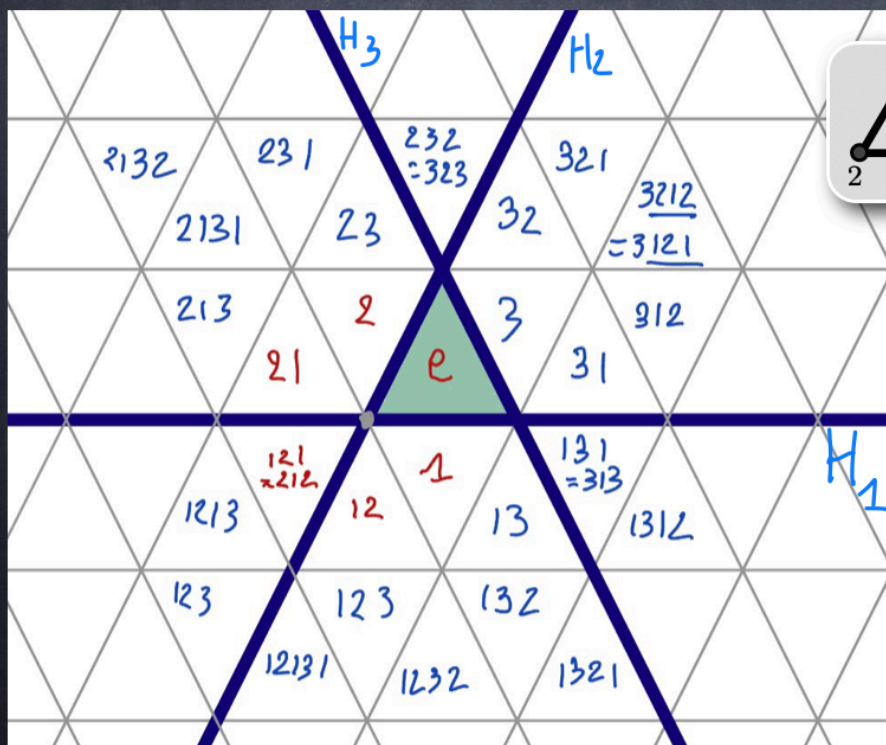
Remark: $s_1 s_2$ is a translation of order $m_{12} = \infty$

$\mathbb{E}$ Euclidean space of dim. $n \in \mathbb{N}^*$

W discrete group generated by Euclidean reflections

T is the set of reflections in W ; $\mathcal{A} = \{H_t \mid t \in T\}$ Coxeter arrangement

Theorem (Coxeter, 1934). There exists $S \subseteq T$ such that (W, S) is a Coxeter system. Moreover $T = \bigcup_{w \in W} wSw^{-1}$.



Example $\mathfrak{A}_2$ ($n = 2$) the affine symmetric group

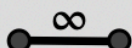
1. Choose $C = e(C)$ a connected comp. of $\mathbb{E} \setminus \bigcup_{t \in T} H_t$ (fund. chamber)
2. S = reflections across wall of C
3. $W \leftrightarrow$ chambers (bijection)

(W, S) is the Coxeter system of type Γ : Coxeter graph Γ :

Theorem (Coxeter, 1934): (W, S) is a Euclidean discrete reflection group iff Γ has spherical or affine types connected components.

Affine types

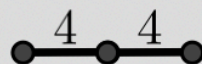
$$I_2(\infty) = \tilde{A}_1$$



$$\tilde{A}_n \ (n \geq 2)$$



$$\tilde{B}_2 = \tilde{C}_2$$



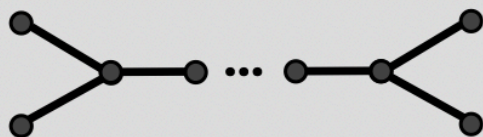
$$\tilde{B}_n$$



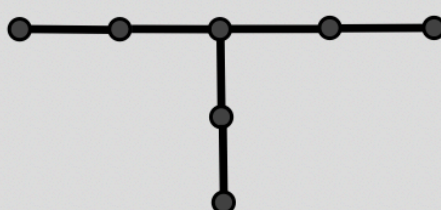
$$\tilde{C}_n$$



$$\tilde{D}_n$$



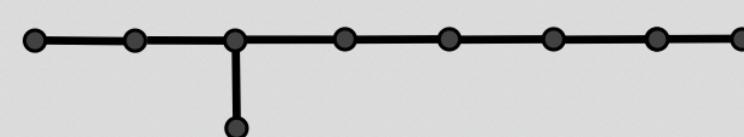
$$\tilde{E}_6$$



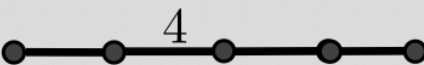
$$\tilde{E}_7$$



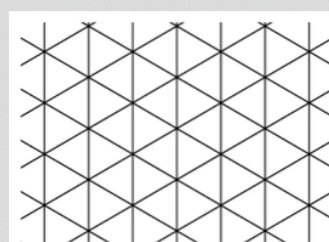
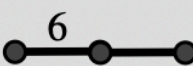
$$\tilde{E}_8$$



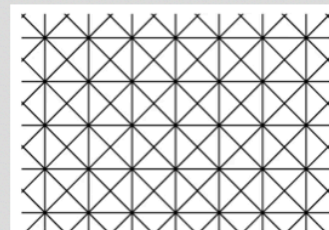
$$\tilde{F}_4$$



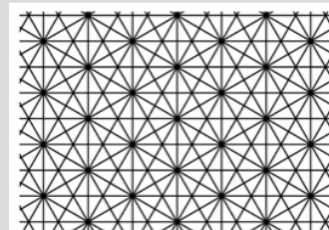
$$\tilde{G}_2$$



$\tilde{A}_2$



$\tilde{C}_2$



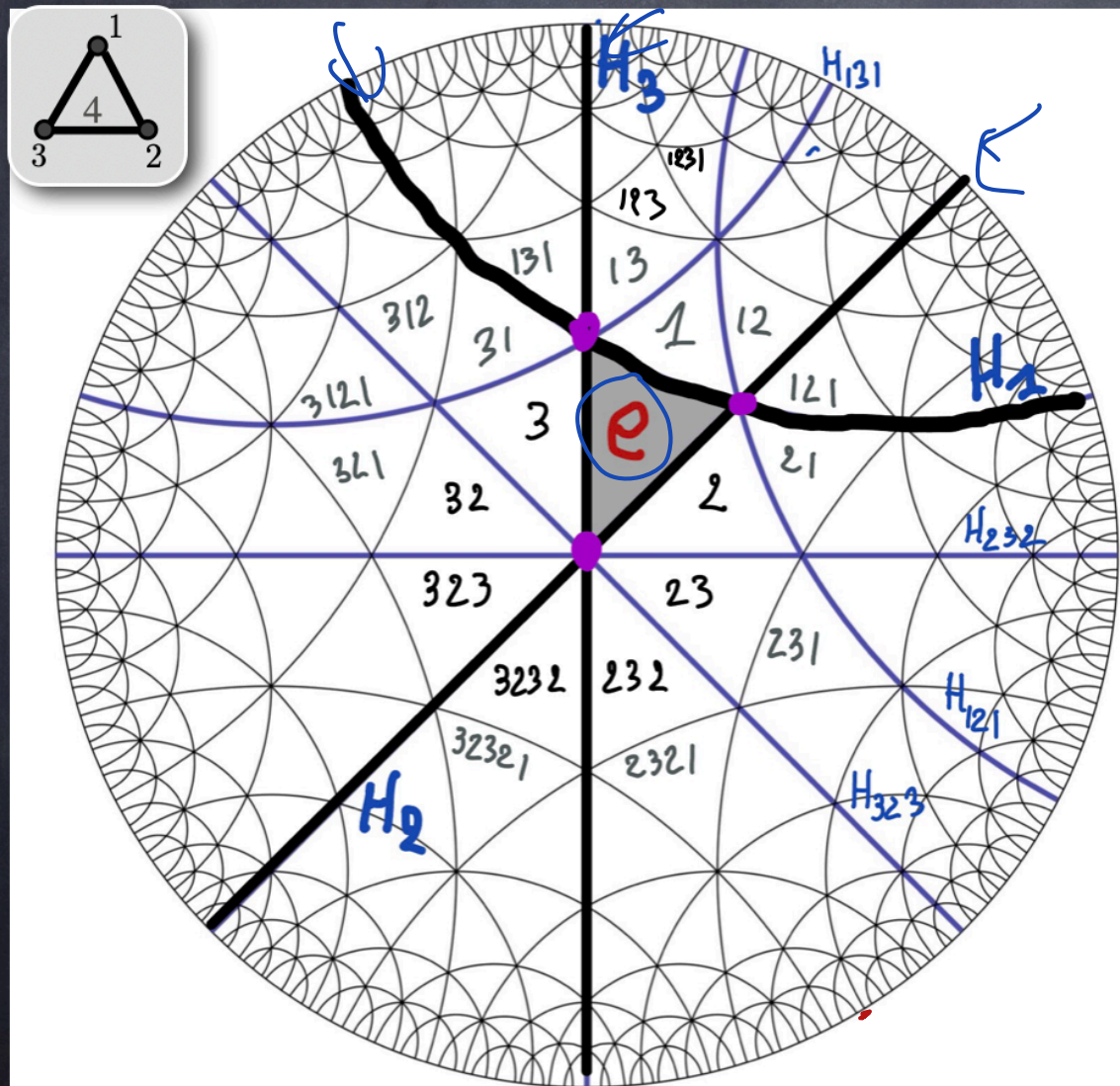
$\tilde{G}_2$

Regular
tilings of
the plane

$\mathbb{H}^n$ hyperbolic space of dim. $n \in \mathbb{N}^*$

W discrete group generated by hyperbolic reflections

T is the set of reflections in W ; $\mathcal{A} = \{H_t \mid t \in T\}$ Coxeter arrangement



Thm (Vinberg 71)
Any discrete group in $\mathbb{H}^n$
Remarks. (W, S) Coxeter syst

1. No complete classifications (in $\mathbb{H}^n$ there are Coxeter systems (W, S) for any $|S| \in \mathbb{N}^*$)
2. Equiv. to classification of Coxeter polytopes (fundamental chamber): for $n = 2$ (Poincaré); $n = 3$ (Andreev's theorem), see works of Felikson, Tamarkin, ...
3. Many conjectures are open (proven in finite and affine case)

$|S| = p$ $\mathbb{H}^n$ all the (W, S) of rank p that are in $\text{Isom}(\mathbb{H}^n)$

DISCRETE REFLECTION GROUPS IN QUADRATIC SPACES

(V, B) real quadratic space of dim n

- V is a $\mathbb{R}$ -vector space and
- $B: V \times V \rightarrow \mathbb{R}$ symmetric bilinear form;
- $q: V \rightarrow \mathbb{R}$ is a quadratic form
 $v \mapsto B(v, v)$

signature:

$$\text{sgn}(B) = (p, q, r)$$

$\uparrow \quad \uparrow \quad \uparrow$
 $+$ $-$ 0
 eigenvalues

The "orthogonal" of $X \subseteq V$ is $X^\perp = \{v \in V \mid B(x, v) = 0 \ \forall x \in X\}$

$\hookrightarrow (V, B)$ is non degenerate if $V^\perp = \{0\}$ (then orthogonal supplement)

The isotropic cone $Q = \{v \in V \mid q(v) = 0\} \supseteq V^\perp$

$\hookrightarrow Q = \{0\}$ then (V, B) Euclidean space inner product B or $-B$.

REFLECTIONS. For $\alpha \in V \setminus Q$, the α -reflection is



$$D_\alpha(v) = v - 2 \frac{B(\alpha, v)}{q(\alpha)} \alpha \quad \text{in } O(V, B)$$

orthogonal group
of (V, B)

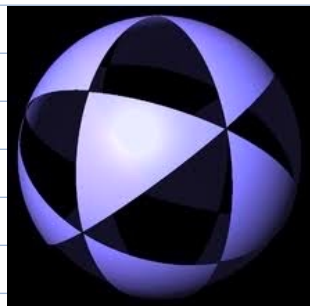
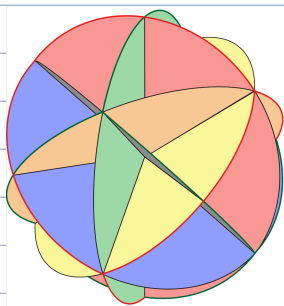
PROPOSITION (EXERCISE): D_α involution and $\text{Fix}(D_\alpha) = \mathbb{R}\alpha^\perp$ an hyperplane

(A)

Spherical geometry

$$\text{sgn}(B) = (n, 0, 0)$$

Euclidean vector space



$$\mathbb{S}^{n-1} \text{ in } V$$

CASES A & C:

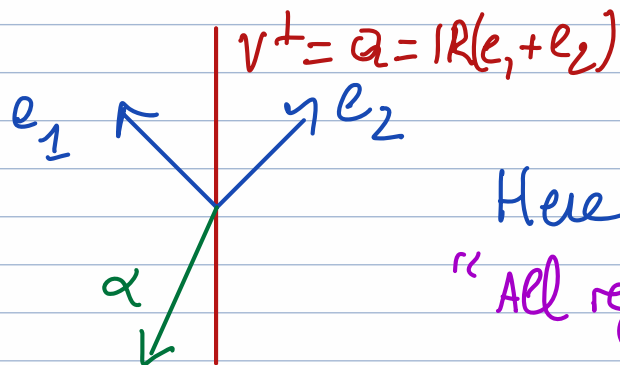
$$\Delta_\alpha = \Delta_\beta \iff \alpha^\perp = \beta^\perp$$

CASE B: FALSE (but true in V^*)

$$V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$$

$$B(e_1, e_2) = -1$$

$$B(e_i, e_i) = 1$$

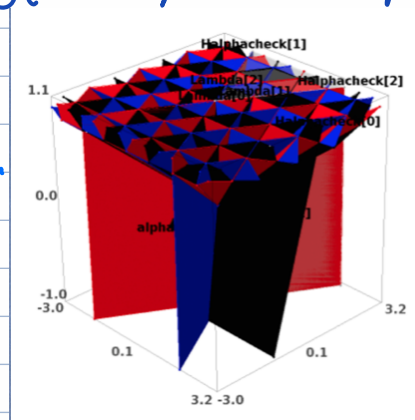


(B)

Euclidean geometry

$$\text{sgn}(B) = (n-1, 0, 1)$$

"Affin" quadratic space



V^*

$$V^\perp = \{v \in V \mid B(v, x) = 0 \forall x \in V\}$$

$$= \mathbb{R}\sigma_0 = \mathbb{Q}$$

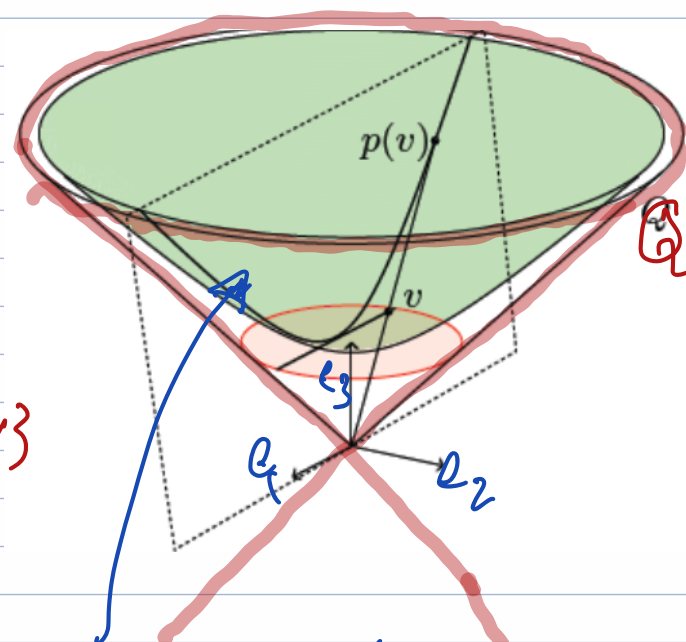
$$(V^\perp)^\perp \simeq V/V^\perp \text{ Euclidean}$$

(C)

Hyperbolic geometry

$$\text{sgn}(B) = (n-1, 1, 0)$$

LORENTZIAN space



$$H^{n-1} = \{v \in V \mid q(v) = -1, \text{ coord } e_3 > 0\}$$

Here $\alpha^\perp = V^\perp$ for all $\alpha \in V$

"All reflections have the same hyperplane"

A simple system Δ in (V, B) is a finite subset $\Delta \subseteq V, \Delta \neq \emptyset$:

- Δ positively linearly independent $(\sum_{\alpha \in \Delta} a_\alpha = 0, a_\alpha \in \mathbb{N} \Rightarrow a_\alpha = 0 \forall \alpha)$

$$\begin{cases} B(\alpha, \alpha) = q(\alpha) = 1 \end{cases}$$

$$\begin{cases} B(\alpha, \beta) \in \left\{ -\cos \frac{\pi}{m} \mid m \in \mathbb{N}_{\geq 2} \right\} \cup]-\infty, -1] \end{cases}$$

$$\Delta_\alpha(\alpha) = -\alpha$$

CONSIDER. $S = \{ \Delta_\alpha \mid \alpha \in \Delta \}$ and $W = \langle S \rangle \leq O(V, B)$

ROOT SYSTEM $\Phi = W(\Delta)$ ("ROOTS ARE NORMAL VECTORS TO REFLECTION HYPERPLANE")

POSITIVE & NEGATIVE ROOTS

$$\Phi^+ = \text{cone}(\Delta) \cap \Phi; \Phi^- = -\Phi^+$$

Theorem (Vinberg 1971)

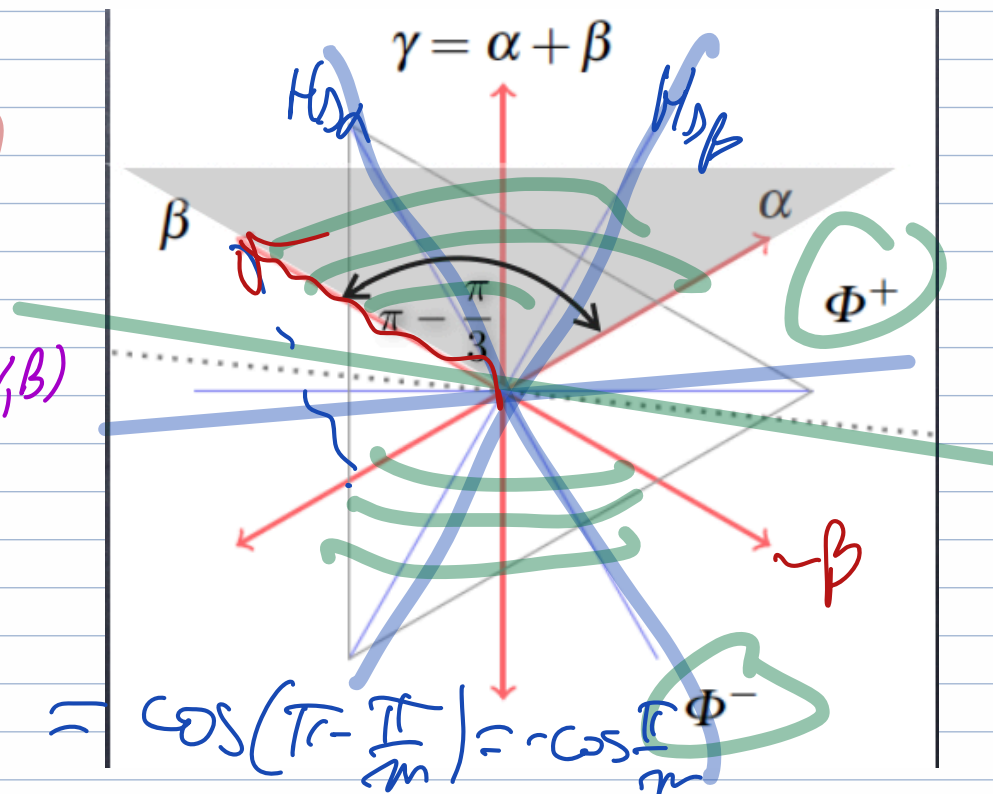
(a) W discrete reflection group in $O(V, B)$

(b) (W, S) Coxeter system

$$(c) \Phi = \Phi^+ \sqcup \Phi^-$$

$$(d) \text{ord}(\Delta_\alpha \Delta_\beta) = m \text{ or } \infty$$

$$B(\alpha, \beta) = \cos\left(\pi - \frac{\pi}{m}\right) = -\cos \frac{\pi}{m}$$



GEOMETRIC REPRESENTATIONS

COXETER SYSTEM (W, S) of TYPE Γ :

$$W = \langle S \mid s^2 = (st)^{m_{st}} = e, \text{ if } m_{st} \neq \infty \rangle$$

Is $m_{st} = \text{ord}(st)$? (we know $\text{ord}(st) \mid m_{st}$)

Def. Let (W, S) be a Coxeter system. A geometric representation of (W, S) is a FAITHFUL REPRESENTATION

$\rho: W \longrightarrow O(V, B)$ given by $\rho(s) = \Delta \alpha_s$ such that:

• $\Delta = \{ \alpha_s \mid s \in S \}$ is a simple system with

$$B(\alpha_s, \alpha_t) = \begin{cases} -\cos \frac{\pi}{m_{st}} & \text{if } m_{st} \in \mathbb{N} \geq 2 \\ \text{a choice of } a_{st} \leq -1 & \text{if } m_{st} = \infty \end{cases}$$

Theorem (Vinberg 71, Tits 68 for $a_{st} = -1$ and Δ basis)

[Geometric representations exist. In part. $\text{ord}(st) = m_{st}$.]

At the core of Theory of Coxeter groups:

REFLECTIONS, ROOTS & TILINGS/COXETER ARRANGEMENT

(W, S) Coxeter system can be faithfully represented in $O(V, B)$

$T = \bigcup_{w \in W} wSw^{-1}$ reflections corresponds to roots in Φ :

$$\epsilon = w\alpha_i w^{-1} \longmapsto w(\alpha_i) = \alpha_\epsilon$$

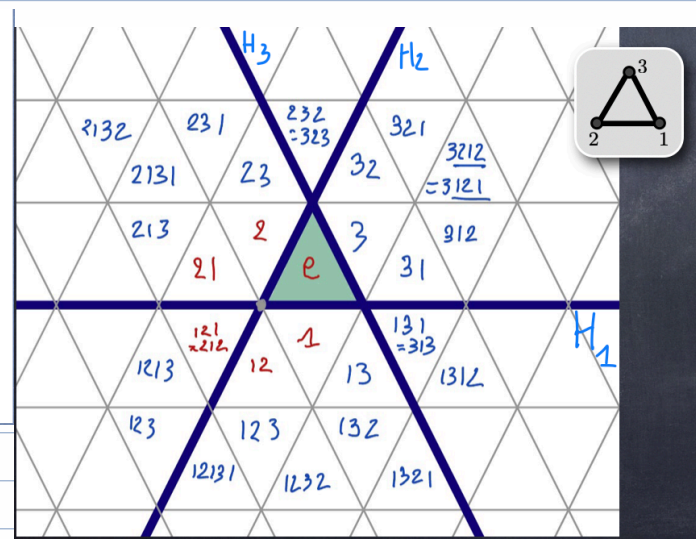
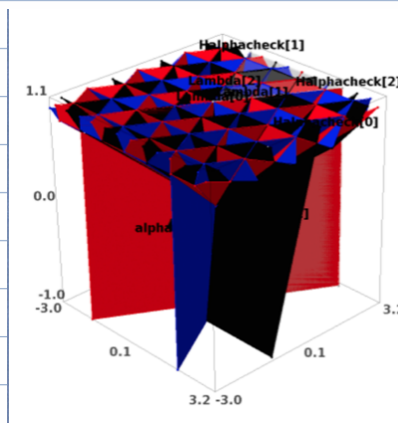
Hyperplanes of reflections are in the dual V^* ($V = V^*$ if $V^\perp = \{0\}$)

Coxeter arrangements and tilings (Tits cone or imaginary cone in V^*)

Fundamental chamber C'

Tiling $\bigcup_{w \in W} w(C') = X$ a cone

$$C = \{ \rho \in V^* \mid \rho(\alpha_i) \geq 0 \forall i \in S \}$$



$$\text{sgn}(B) = (2, 1, 0)$$

At the core of Theory of Coxeter groups:

REFLECTIONS, ROOTS & TILINGS/COXETER ARRANGEMENT

(W, S) Coxeter system can be faithfully represented in $O(V, B)$

$T = \bigcup_{w \in W} wSw^{-1}$ reflections corresponds to roots in Φ :

$$H_E \subseteq X; H_E = \{p \in X \mid p(\alpha_E) = 0\}$$

$E = w\Delta w^{-1} \mapsto w(\alpha_\Delta)$

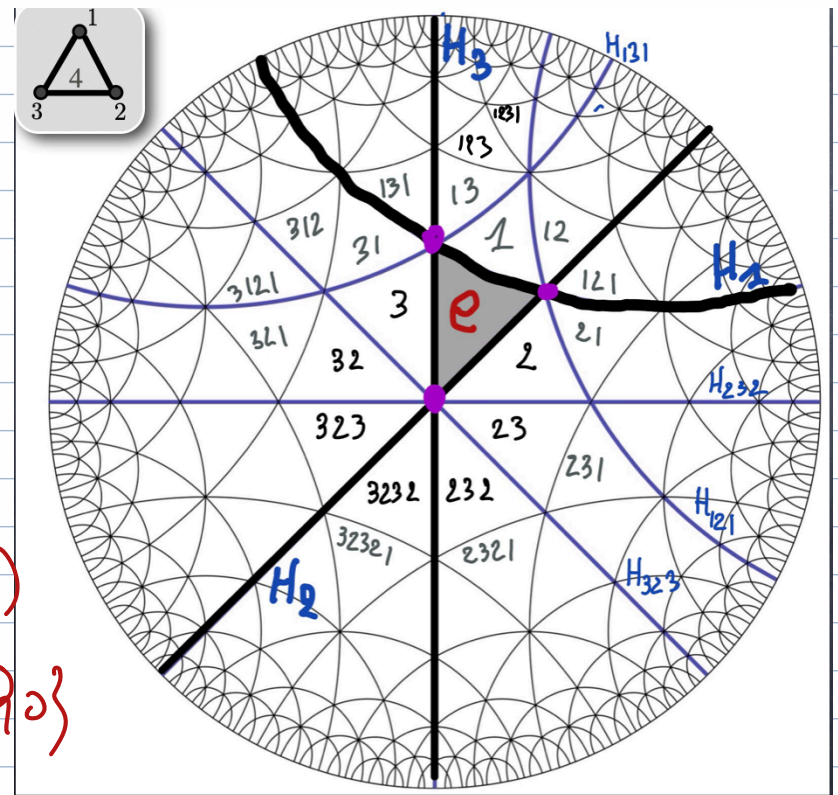
Hyperplanes of reflections are in the dual V^* ($V = V^*$ if $V^\perp = \{0\}$)

Coxeter arrangements and tilings (Tits cone or imaginary cone in V^*)

Fundamental chamber C'

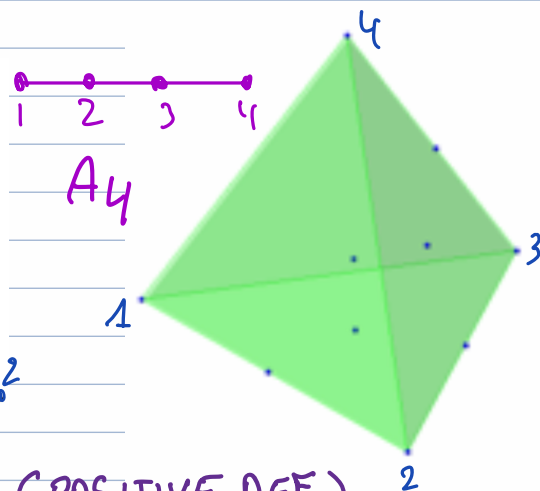
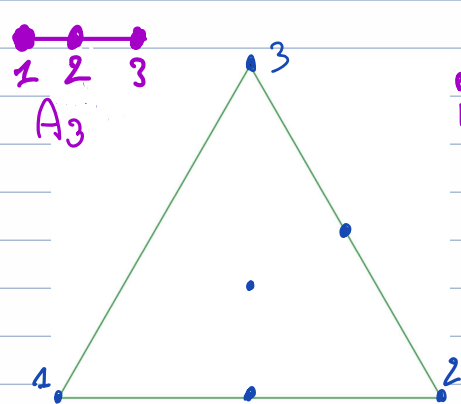
Tiling $\bigcup_{w \in W} w(C') = X$ a cone

$\text{Sgn}(\beta) = (2, 1, 0)$
 $V^\perp = \{0\}$



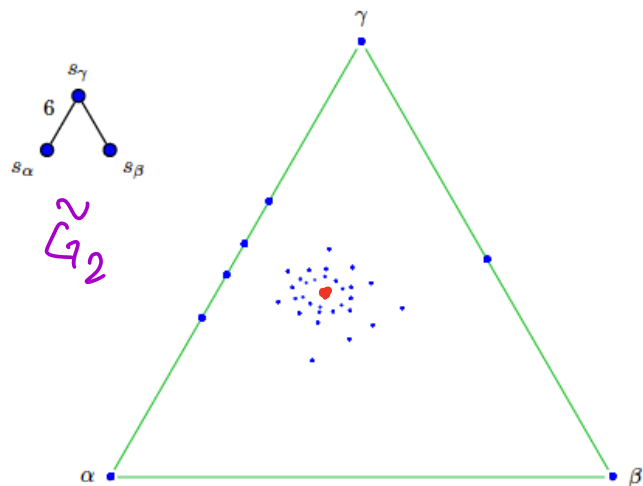
THREE TYPES OF COXETER SYSTEMS

(depend of the signature of a geom. representation)



FINITE TYPE (POSITIVE DEF.)

$$\text{Acc}(IP\Phi) = \emptyset$$

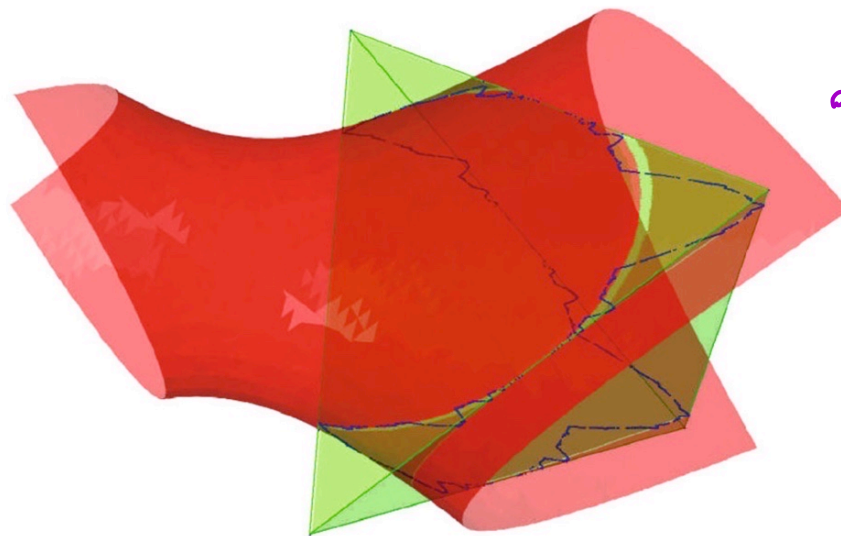
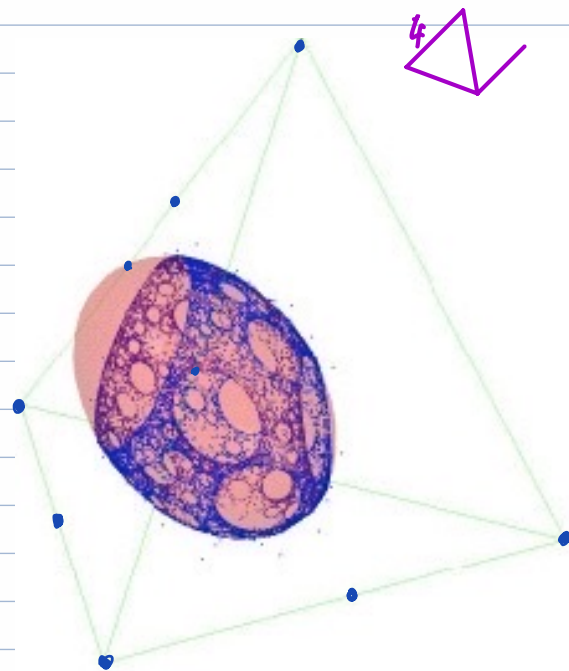


$$\text{Sgn}(2, 2)$$

AFFINE TYPE (Pos. SEMI. DEF)

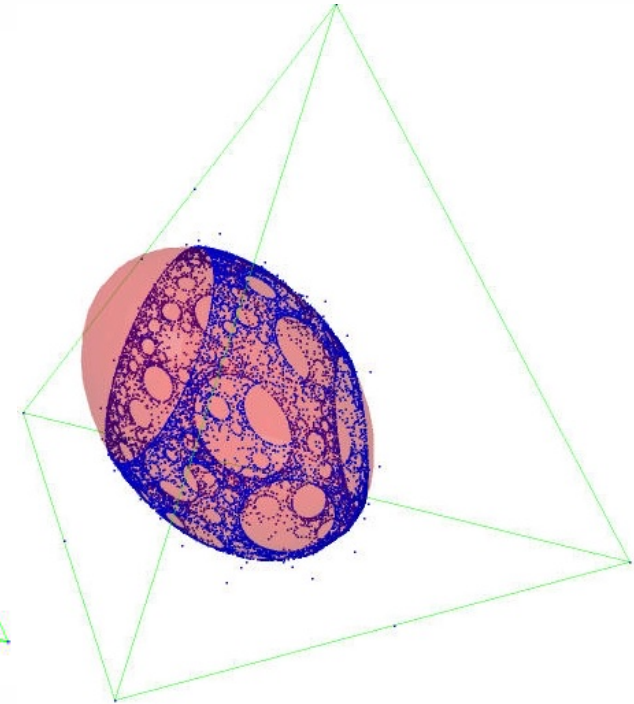
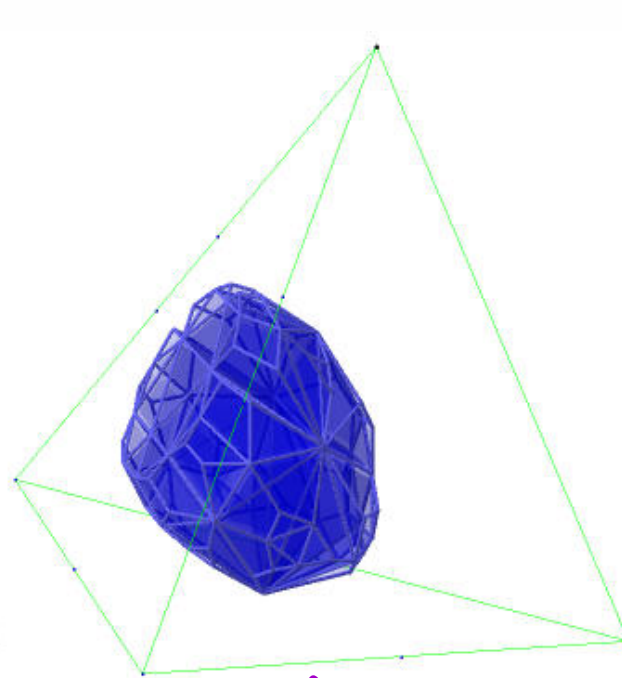
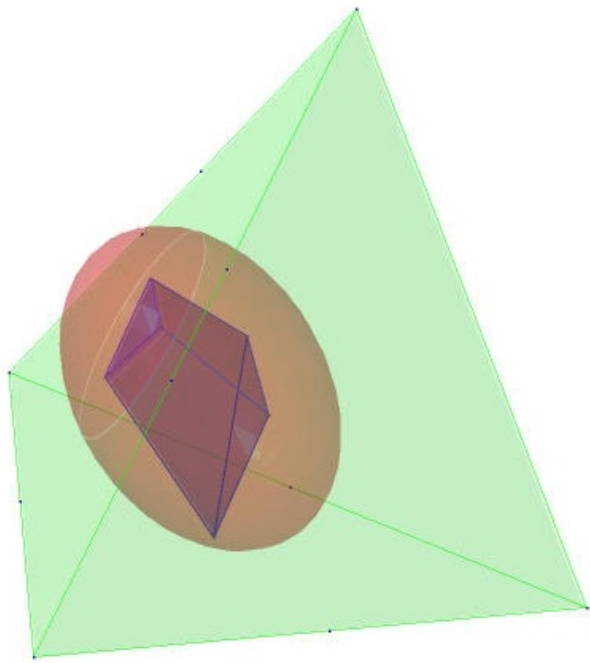
$$\text{Acc}(IP\Phi) \text{ FINITE}$$

Hypobolic
INDEFINITE
TYPE
 $\text{Acc}(IP\Phi)$
INFINITE

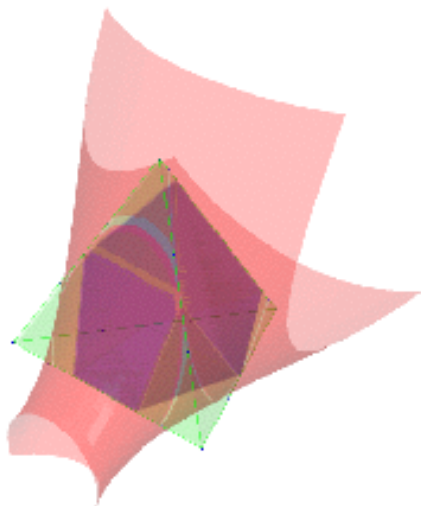


(cf Dyer-CH-RIPOU ; CH-RIPOU-LABBÉ ≈ 2013)

IMAGINARY CONES & TILINGS (Dyer, CH, RIPPOLL 2013)



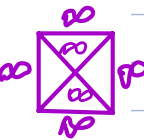
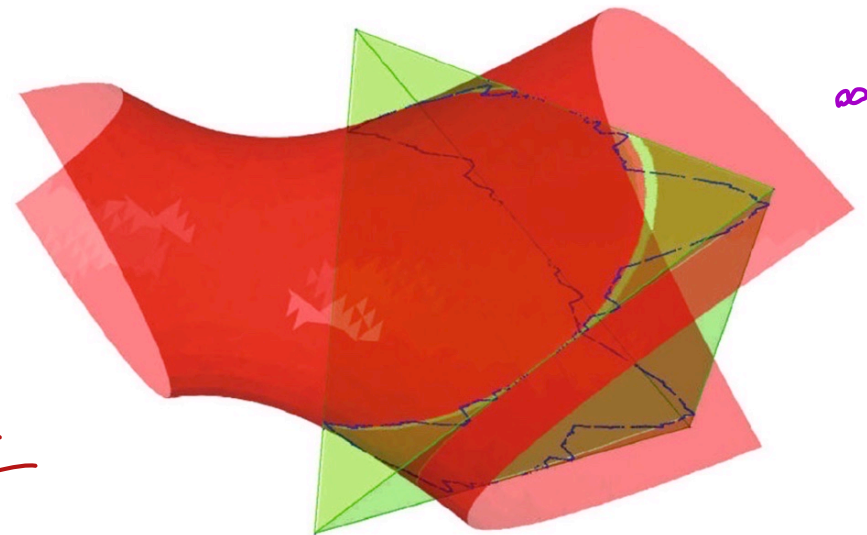
Hyperbolic



Sg_4

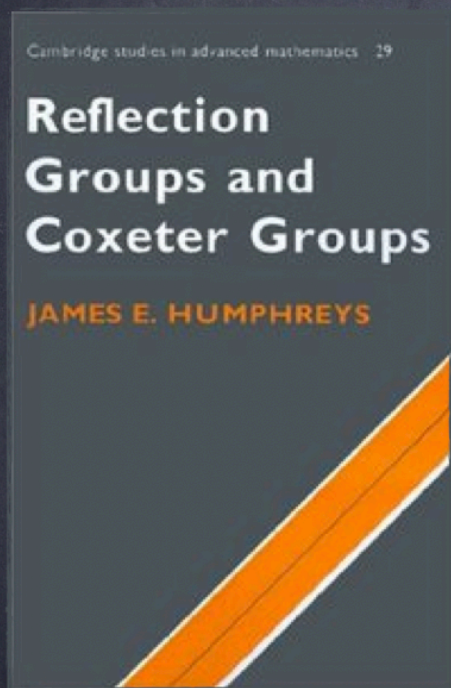
$(2,2)$

NOT HYPERBOLIC

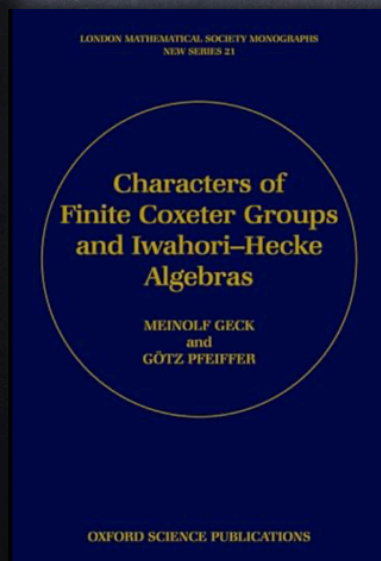


SOME SELECTED BOOKS

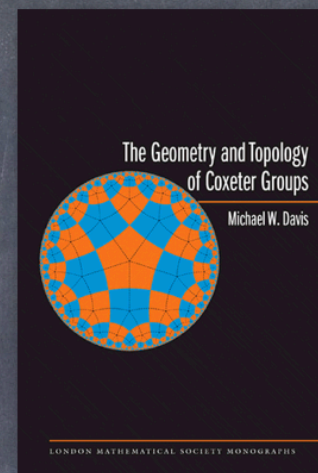
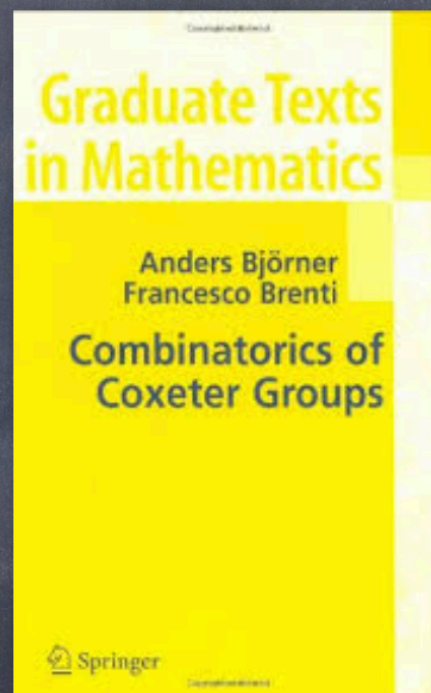
1990



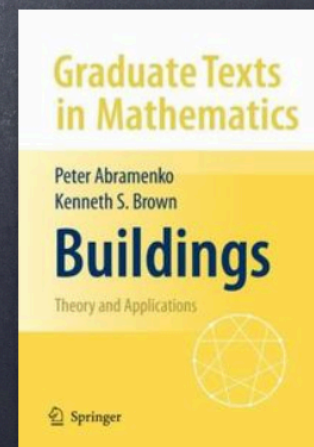
2003 – finite Coxeter groups



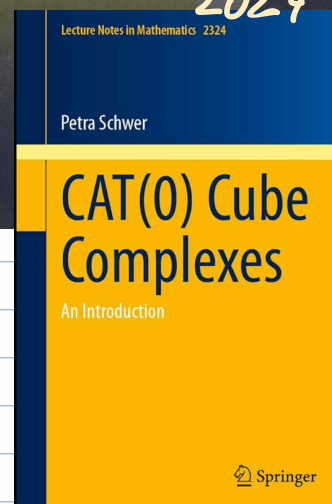
2005



2008



2024



LIE THEORY, BUILDINGS,
Alg. groups, Cluster alg, ...

Coxeter groups and Lie theory

(a quick overview)

$\tilde{\Phi}$ is a crystallographic root system in a Euclidean space

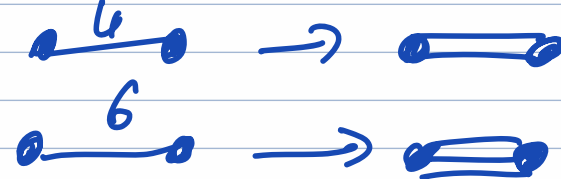
• $\Phi = \{ \frac{\alpha}{\|\alpha\|} \mid \alpha \in \tilde{\Phi} \}$ is a root system

• $\tilde{\Phi}$ stable under W

• $\frac{2B(\alpha, \beta)}{\|\alpha\|^2} \in \mathbb{Z}$



Dynkin diagram



W is then called a Weyl group: $m_{\alpha} \in \{2, 3, 4, 6\}$ only.

$\mathfrak{g}$ semisimple Lie algebra over $\mathbb{C}$

$\mathfrak{h}$ Cartan subalgebra

Then $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \tilde{\Phi}} \mathfrak{g}_{\alpha}$, $\mathfrak{g}_{\alpha} = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$

II How to work with Coxeter Systems.

a) Characterizations.

(W, S) Involution system: $W = \langle S \rangle$ group; $\text{ord}(s) = 2$ for all $s \in S$.

Cayley graph $\text{Cay}(W, S)$: vertices W , edges  $w \in W, \delta \in S$.

Length $l: W \rightarrow \mathbb{N}$,

$w \mapsto l(w) = \text{length of a geodesic from } e \text{ to } w \text{ in } \text{Cay}(U/S)$
 $= \text{length of a reduced word .}$

MetRIC on $\text{Coxy}(W, S)$: $d(u, v) = \ell(u^{-1}v)$. (exercise)

- For all $w \in W, s \in S$, we have:

$$(i) \ell(w) - 1 \leq \ell(ws) \leq \ell(w) + 1.$$

case $l(ws) = l(w)$ possible!

(ii) $\ell(\omega s) = \ell(\omega) \pm 1 \iff \exists \varepsilon: W \rightarrow \{\pm 1\}$ gpe homomorphism (sign character)
 $(\forall w \in W, \forall s \in S) \quad s \mapsto -1$

Corollary: (W.S) Coxeter system has a signature and $\varepsilon(w) = (-1)^{\ell(w)}$

• Weak order

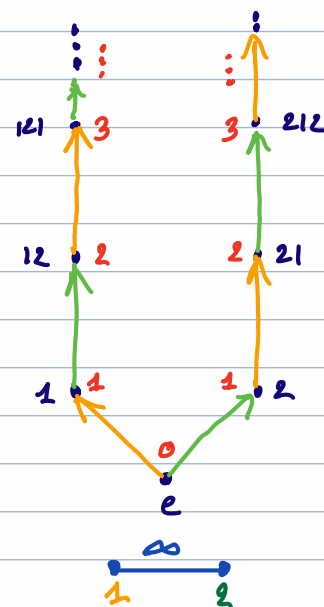
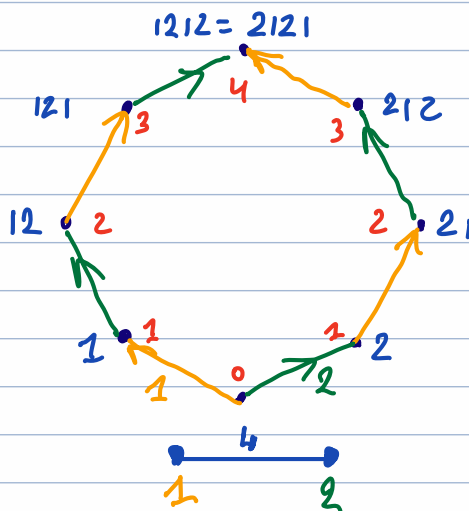
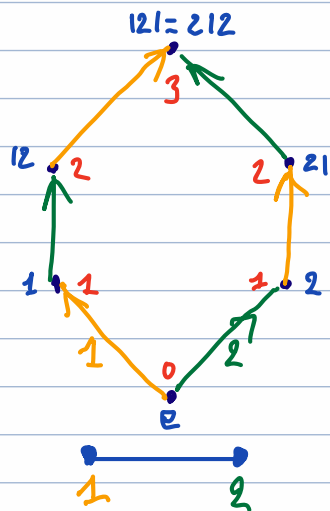
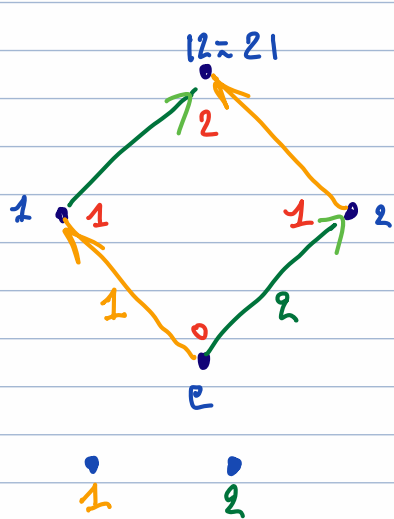
(Right) weak order $(W, \leq_R)$: we orient the Cayley graph $\text{Cay}(W, S)$

Hasse diagram $\overrightarrow{\text{Cay}(W, S)}$ $w \xrightarrow{s} w_\Delta$ if $\ell(w) = \ell(w_\Delta) + 1$.

$u \leq_R w$ iff there is a geodesic from e to w passing through u (u prefix)

$w = \underbrace{uv}_{\text{prefix}}$ reduced

Cayley graph "oriented": weak order $(W, \leq_R)$ $\ell(w)$ "in red"

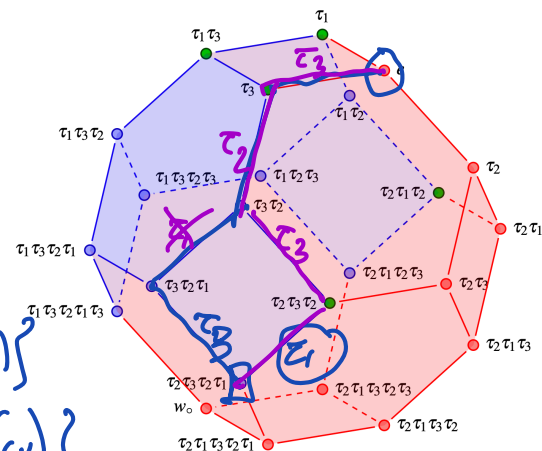


$$(D_1 D_2)^3 = e \iff D_1 D_2 D_1 (D_2 D_1 D_2) = e$$

For (W, S) an involution system, we consider:

• $T = \bigcup_{w \in W} wSw^{-1}$ reflections;

• Left and Right descent sets: $D_L(w) = \{s \in S \mid \ell(sw) < \ell(w)\}$
 $D_R(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$



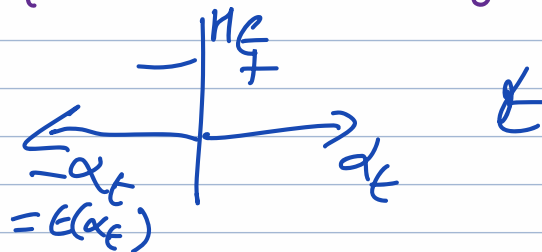
Permutahedron

Theorem. Let (W, S) be an involution system. Then the following are equivalent.

(i) (W, S) is a Coxeter system

(ii) There is an action of W on $T \times \{\pm 1\}$ such that for all $s \in S$

$$s \cdot (t, \varepsilon) = \begin{cases} (sts, \varepsilon) & \text{if } s \neq t \\ (s, -\varepsilon) & \text{if } s = t \end{cases}$$



(iii) (W, S) satisfies the exchange reduction condition (illustrate with geodesic)

for any $w = \underline{s_1 \dots s_k}$, $s_i \in S$, and $s \in D_R(w)$, $\exists 1 \leq i \leq k$ s.t: $w = \underline{s_1 \dots s_i} \dots \underline{s_k s}$

(iv) (W, S) satisfies the braid monoid condition. $\rightarrow$ Braid group
 $\oplus$ sign character

Braid monoid condition

(W, S) Involution system. For $s, t \in S$, $m_{st} := \text{ord}_W(st)$

Braid relations: $(st)^{m_{st}} = e$ $\iff$ $\underbrace{stst \dots}_{m_{st} \text{ letters}} = \underbrace{tst \dots}_{m_{st} \text{ letters}}$

$m_{st} \neq \infty$ $\iff$ $s^2 = e$ $t^2 = e$

braid relation

Braid monoid condition (BM): (W, S) satisfies BM if

for any monoid $\mathcal{M}$ and map $f: S \rightarrow \mathcal{M}$ such that

$$\underbrace{f(s)f(t)f(s) \dots}_{m_{st} \text{ letters}} = \underbrace{f(t)f(s)f(t) \dots}_{m_{st} \text{ letters}}, \text{ for all } s, t \in S \text{ with } m_{st} \neq \infty.$$

there is a well-defined map $\bar{f}: W \rightarrow \mathcal{M}$

$$\underbrace{w = s_1 \dots s_k}_{\text{reduced word}} \mapsto f(s_1) \dots f(s_k)$$

Consequence: any reduced words for $w \in W$ are equivalent up to braid relations (Matsumoto theorem)

. Inversion sets

(W, S) Coxeter system:

Reflections of (W, S) : $T = \bigcup_{w \in W} w S w^{-1} \quad (S \subseteq T)$

Inversion set of $w \in W$: $T(w) = \{t \in T \mid \ell(tw) < \ell(w)\}$

Proposition (exercise): $\ell(w) = |T(w)|$

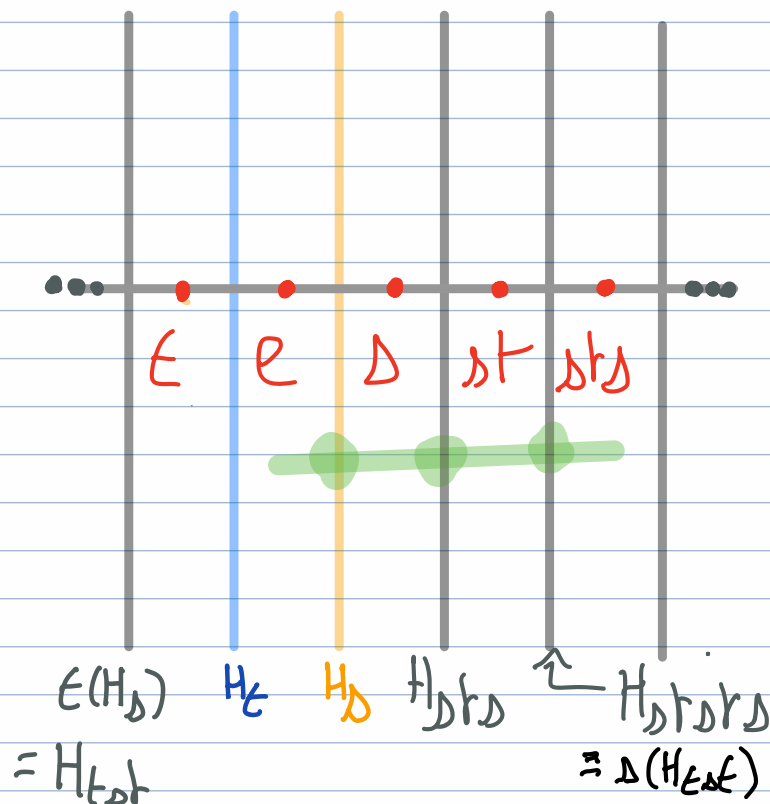
$u \leq_R v \iff T(u) \subseteq T(v)$
 $\quad \quad \quad \nwarrow$ Right weak order

Example. D_∞ 

$T = \{\delta, \epsilon, \delta\epsilon, \epsilon\delta, \delta\epsilon\delta, \dots\}$

$T(\delta\epsilon\delta) = \{\delta, \delta\epsilon, \delta\epsilon\delta\}$
 $= \{t \in T \mid H_t \text{ separates } e \text{ from } \delta\epsilon\}$

$\ell(\delta\epsilon\delta) = 3 = |T(\delta\epsilon\delta)|$



Reflections of (W, S) : $T = \bigcup_{w \in W} w S w^{-1}$ ($S \subseteq T$)

Coxeter (hyperplane) arrangement: $\mathcal{A}_W = \{H_\ell \mid \ell \in T\}$ (in some space X)

$W \xleftrightarrow{1:1}$ Chambers (regions of $X \setminus \mathcal{A}_W$)

Proposition: $T(w) = \{\ell \in T \mid H_\ell \text{ separates } e \text{ from } w\}$.

Example: $\mathfrak{S}_4$ of type A_3

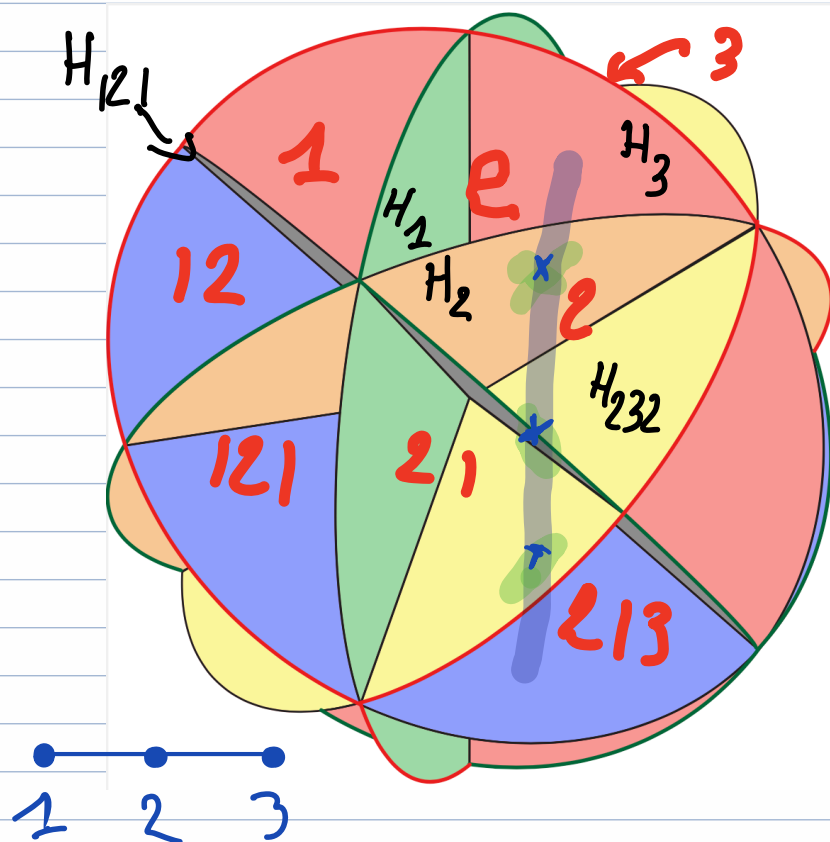
$T(213) = \{2, 232, 121\}$

$\ell(213) = 3$

With root system

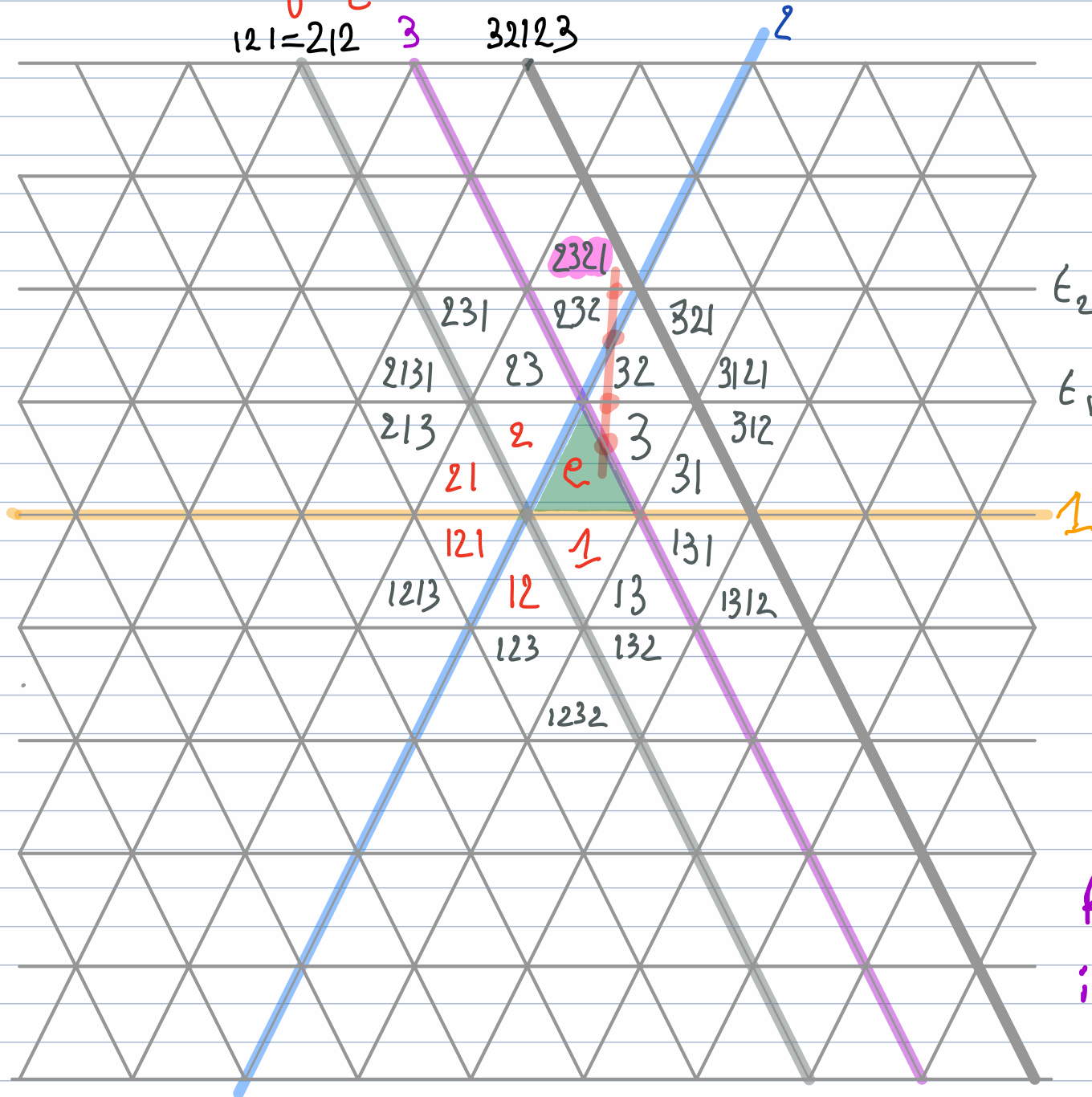
$\Phi(w) = \Phi^+ \cap w(\Phi^-)$

and $T(w) = \{\ell \in T \mid \alpha_\ell \in \Phi(w)\}$



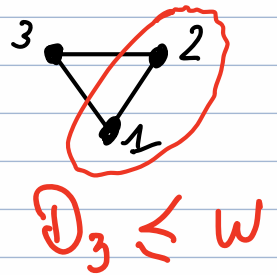
• Coxeter groups as reflection groups

write ϵ instead of H_ϵ



Coxeter arrangement
of w for

(W, S) affine Coxeter
system of type $\tilde{A}_2$

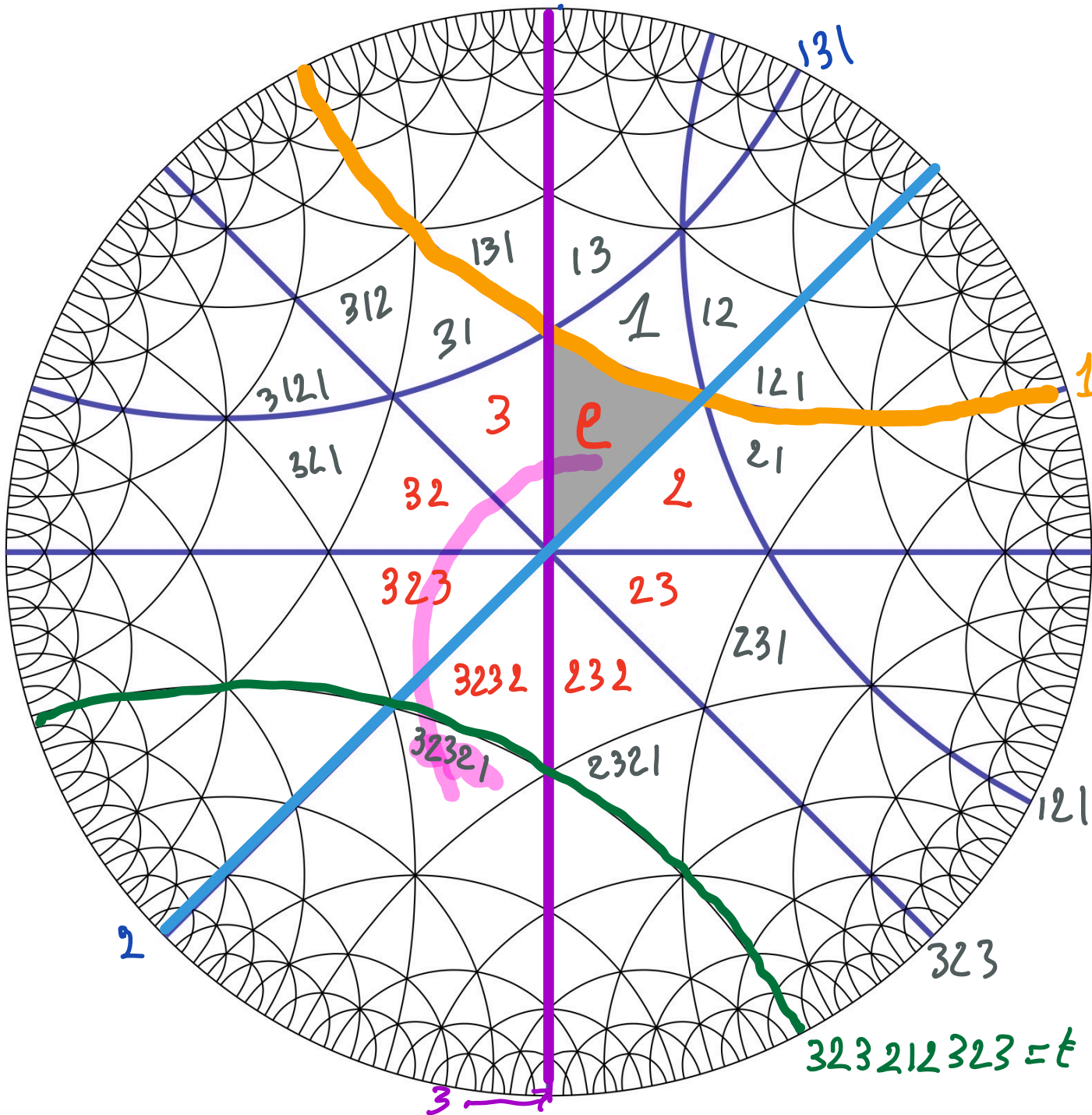


$$T(2321) = \{s_2, s_3, t_1, t_2\}$$

$$l(2321) = 4$$

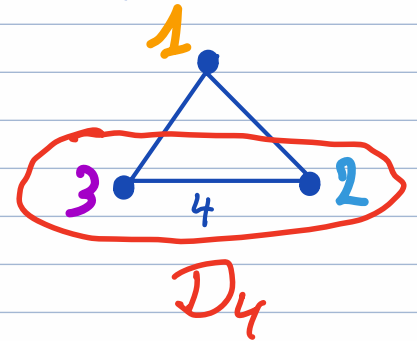
Rk. $H_\epsilon // H_\alpha$ (parallel)
if $t=2$ or $\langle \alpha, \epsilon \rangle = \mathbb{Z}$

- Coxeter groups as reflection groups



Coxeter arrangement
atw for

(W, S) of type



232

$$\tau(32321)$$

$$= \{2, 3, 323, 232, E\}$$

$$\ell(32321) = 5$$

$$Rk. H_t // H_n \text{ (parallel)}$$

$$\text{if } t=n \text{ or } \langle R, t \rangle \in \mathcal{I}_\infty$$

$$323212323 = t$$

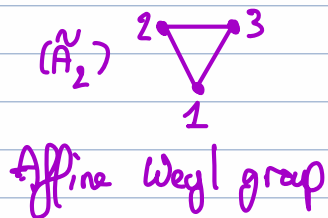
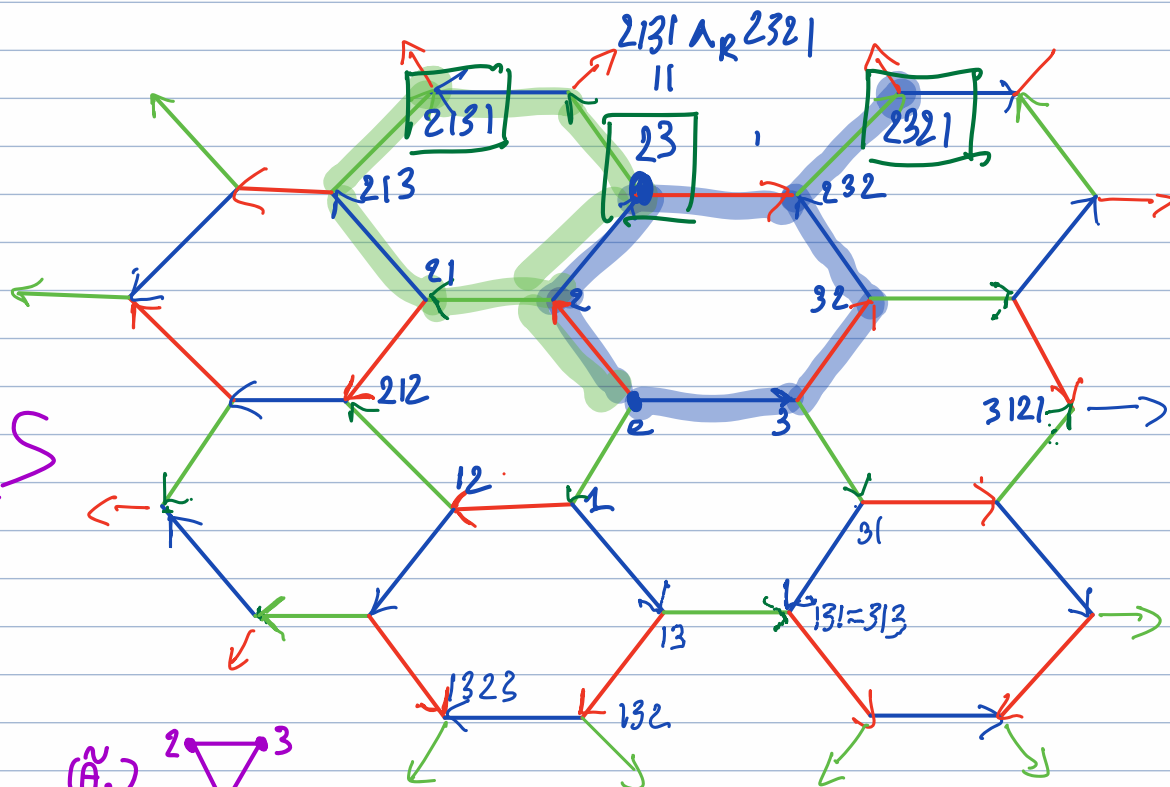
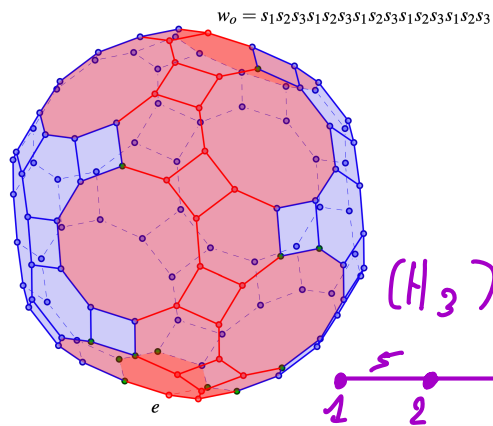
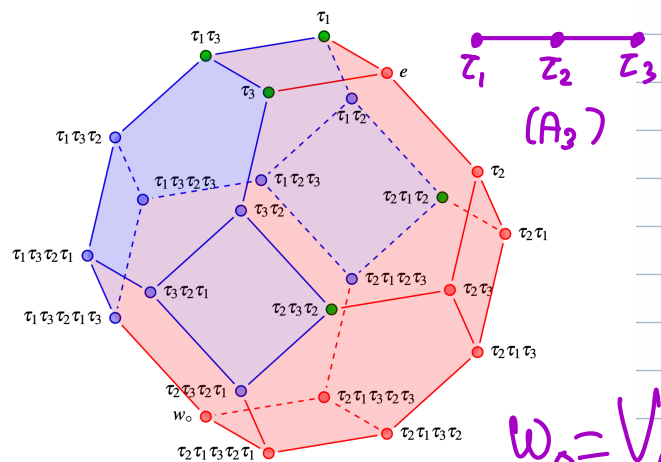
• Weak order is a semilattice

Theorem (Björner 1984) $(W, \leq_R)$ is a complete meet semi-lattice

$\boxed{\forall X \subseteq W, X \neq \emptyset, \text{ the meet } \bigwedge_R X = \sup \{g \in W \mid g \leq_R x, \forall x \in X\} \text{ exists.}}$

Corollary. (a) The join $\bigvee_R X$ of bounded X exists.

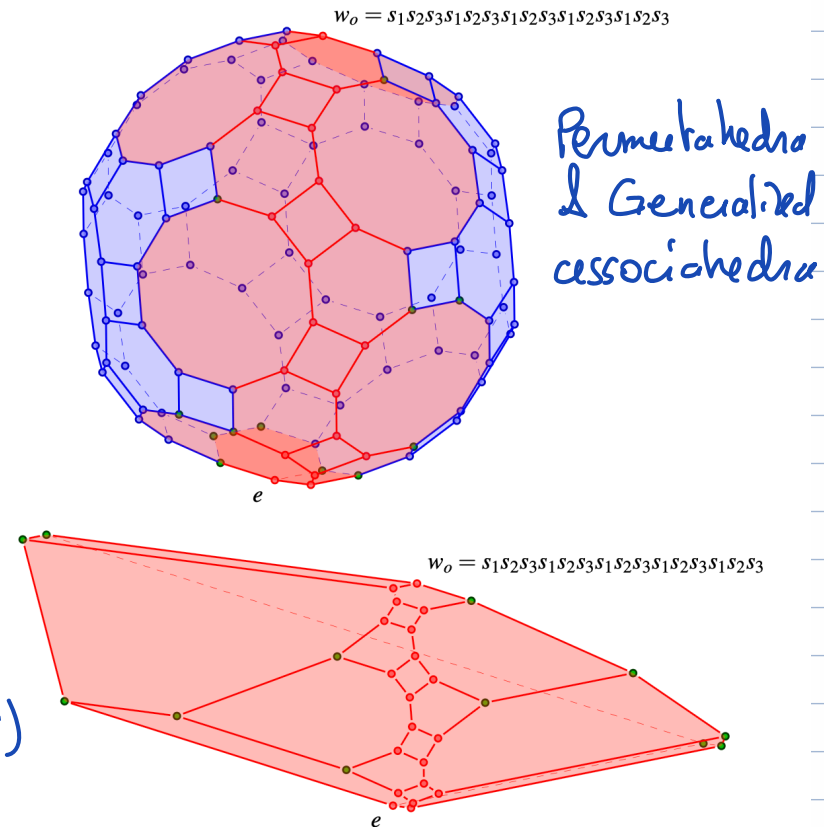
(b) W finite $\iff S$ bounded. In this case $w_0 = \bigvee_R S$ longest elt.



• WHY CONSIDERING THIS PROPERTY?

SELECTED CONSEQUENCES FOR FINITE COXETER SYSTEMS

- Lattice quotients, Cambrian fans, generalized associahedra
g-vector fans of finite type Cluster algebra
(Reading, Reading-Speyer, CH-Lange-Thomas
Stella, Pilaud-Stump ...)
- Lattice theory of torsion classes &
Shards theory in representation theory.
(Demonet-Iyama-Reading-Reiten-Thomas, ...)
- Reflection ordering and initial section
 - ↳ Kazhdan-Lusztig polynomials (Dyer, Brenti)
 - ↳ PBW root vectors (Lusztig ...)



Permutahedra
& Generalized
associahedra

(H_3)



Bruhat Order

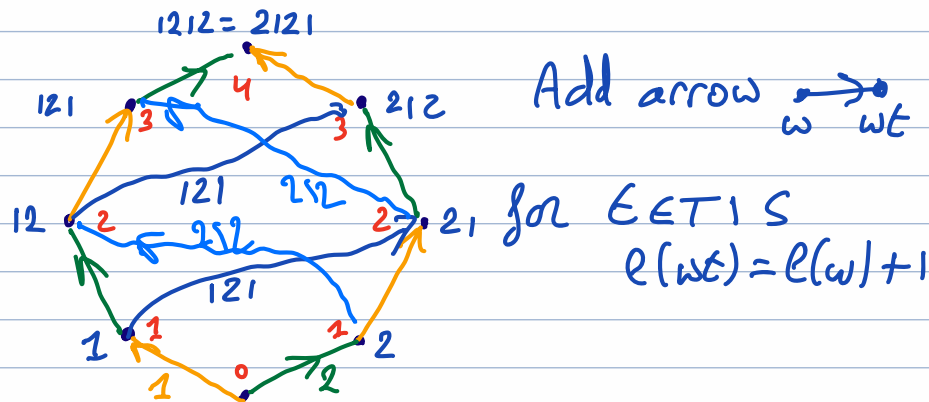
Consider the following directed graph:

vertices W

edges $w \xrightarrow{e} wt$ if $\ell(wt) = \ell(w) + 1$, $w \in W, e \in T$

Proposition. This graph is the Hasse diagram of a poset $(W, \leq)$ called the Bruhat order. The right weak order $(W, \leq_r)$ is a subposet.

Ex. (W, S) of type A_2



Thm. $u \leq v$ iff for any reduced word $v = s_{i_1} \dots s_{i_k}$ there is $1 \leq i_1 < \dots < i_p \leq k$ and $u = s_{i_1} \dots s_{i_p}$.

Motivation: Kazhdan-Lusztig th; Schubert calculus, see Patrik's talk.

The Bruhat graph is obtained by adding moreover all arrows $w \rightarrow wt$ if $\ell(wt) > \ell(w)$ for $w \in W, t \in T$.

In the example above add $\xrightarrow{121} 2121$; $\xrightarrow{212} 121$; $\xleftarrow{121} 121$
 $\xleftarrow{212} 212$

Conjecture (Dyer). $T(uv_r v) = \{t \in T \mid t \text{ is a vertex in a } (u, v)\text{-path in the Bruhat graph}\}$

A (u, v) -path is a path in the Bruhat graph starting at e and labelled by elements in $T(u) \cup T(v)$

Ex: $u = 12, v = 32$

$T(u) = \{1, 121\}$; $T(v) = \{3, 323\}$

So the (u, v) -path have the following

vertices in T : $1, 3, 232, 121, \underline{12321}$,
 NEW

This is T so $T(uv_r v) = T$

and $uv_r v = w_0$

