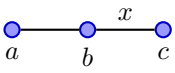
LIST OF EXERCICES FOR OCTOBER 19TH,
MAT995Q–COXETER GROUPS
UQAM 2026

One can use the result of an exercice appearing before the exercise one is solving.
Exercices with $*$ are more difficult.

1. On involution systems and Coxeter systems

Exercise 1.1 (from Björner-Brenti)**.** Let (W, S) be a Coxeter system with $S = \{a, b, c\}$ of type:

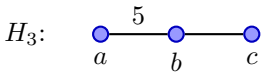


(a) Assume that $bcbcacababcacbcabacbabacbc = e$. Find x.
(b) What is the Coxeter matrix of W ?

Exercise 1.2. Let (W, S) be the involution system presented by:

$$W = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^5 = (bc)^3 = (ac)^2 = e,\ cbabcab = ababacba \rangle.$$

(a) Show that (W, S) is of Coxeter type



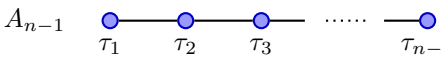
Denote by $(\overline{W}, S)$ the Coxeter system of type H_3. Then we know from Coxeter's classification that $\overline{W}$ is the isometry group of the dodecahedron and is of order 120.
(b) Show that $w = cbabcababcababa$ is an involution in $\overline{W}$ by using only braid relations.
(c) Show that w is in the kernel of the group epimorphism from $\phi : \overline{W} \to W$ mapping arising fro the identity map on S.
(d) Show that W is isomorphic to the alternate group $\mathcal{A}_5$, which is generated by the permutations (written in cycle notation):

$$s = (12)(34),\ t = (23)(45) \text{ and } r = (14)(23).$$

(e) Show that (W, S) do not admit a sign character.

Exercise 1.3 (Symmetric groups are Coxeter groups *)**.** For $n \in \mathbb{N}^*$, we denote by $\mathfrak{S}_n$ the symmetric group of permutations of the set $\{1, \ldots, n\}$ and by $S_{n-1} = \{\tau_1, \ldots, \tau_{n-1}\}$ the set of simple transpositions $\tau_i = (i\ i+1)$. It is well-known that $\mathfrak{S}_n$ is generated by S_{n-1}.

The aim of this exercise is to show that $(\mathfrak{S}_n, S_{n-1})$ *is a Coxeter system of type:*



That is: $\mathfrak{S}_n = \langle\, \tau_i \mid \tau_i^2 = (\tau_i\tau_{i+1})^3 = (\tau_i\tau_j)^2 = e,\ 1 \le i, j \le n-1,\ |i-j| > 1 \,\rangle$.

To avoid confusion, we consider a set $\mathcal{S}_{n-1} = \{s_1, \cdots, s_{n-1}\}$ in bijection with S_{n-1} and we denote by $\mathcal{F}_{\mathcal{S}_{n-1}}$ the free group over $\mathcal{S}_{n-1}$. Consider the Coxeter matrix m over $\mathcal{S}_{n-1}$ defined by $m(s_i, s_j) = \mathrm{ord}_{\mathfrak{S}_n}(\tau_i\tau_j)$ for all $1 \le i \le j \le n-1$. We consider the group:

$$G_n = \mathcal{F}_{\mathcal{S}_{n-1}}/N(R_m),$$

where $R_m = \{(st)^{m(s,t)} = e \text{ for } s, t \in \mathcal{S}_{n-1}\} \subseteq \mathcal{F}_{\mathcal{S}_{n-1}}$.

(a) Let $n \in \mathbb{N}_{\ge 3}$ and consider $H_{n-1} = \langle\, \mathcal{S}_{n-2} \,\rangle = \langle\, s_1, \ldots, s_{n-2} \,\rangle \le G_n$.
  (i) Show that there is a unique group epimorphism $f_{n-1} : G_{n-1} \to H_{n-1}$ such that $f_{n-1}(s) = s$ for all $s \in \mathcal{S}_{n-2}$.

(ii) Consider the following elements in G_n:

$$y_0 = e,\ y_1 = s_{n-1},\ y_i = s_{n-i}\cdots s_{n-1} \quad (1 \le i \le n-1).$$

Show that G_n is the union of the cosets $y_i H_{n-1}$.

(b) Show by induction on $n \in \mathbb{N}^*$ that $|G_n| \le n!$

(c) Show that $(\mathfrak{S}_n, S_{n-1})$ is a Coxeter system of type A_{n-1} for all $n \in \mathbb{N}^*$.

(d) Deduce that $H_{n-1} \simeq G_{n-1}$ and that $(H_{n-1}, \mathcal{S}_{n-2})$ is a Coxeter system of type A_{n-1}, for all $n \in \mathbb{N}_{\ge 3}$.

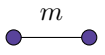
2. On dihedral groups

In the linear Euclidean plane, i.e., a vector space of dimension 2 with a scalar product, we consider a rotation r of order $m \in \mathbb{N}_{\ge 2}$ and a reflection s. The dihedral group $\mathcal{D}_m$ (of order $2m$) is the group generated by r and s. If we set $t = sr$, then t is a reflection and:

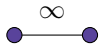
$$\mathcal{D}_m = \langle\, s, t \mid s^2 = t^2 = (st)^m = e \,\rangle.$$

In the Euclidean plane, let H be a line, we consider a translation u (of order ∞) along a vector perpendicular to H and a reflection s through H. The dihedral group $\mathcal{D}_\infty$ is the infinite group generated by u and s. Set $t = us$, then t is a reflection and $\mathcal{D}_\infty = \langle\, s, r \mid s^2 = t^2 = e \,\rangle$..

Exercise 2.1 (Groups generated by two involutions are dihedral groups)**.** Let G be a group generated by a set of two distinct involutions $A = \{a, b\}$.

(a) Assume that G is finite, show that the cardinality of G is $|G| = 2m$ for some integer $m \ge 2$. Show that (G, A) is a Coxeter system of type 

.

(b) If G is infinite, show that (G, A) is a Coxeter system of type 

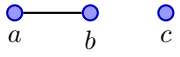
.

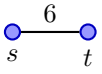
Exercise 2.2. Let (V, B) be a real quadratic plane with basis $\Delta = \{\alpha_1, \alpha_2\}$ and the matrix of B in the basis Δ is:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

For $i = 1, 2$, denote by $s_i = s_{\alpha_i}$ the reflection associated to the vector α_i and $G = \langle s_1, s_2 \rangle$ the reflection subgroup of $O(V, B)$ generated by s_1 and s_2. The *root system of* G is the orbit $\Phi = G(\Delta)$. Show that $\Phi = \pm\{n\alpha_1 + (n+1)\alpha_2, n\alpha_2 + (n+1)\alpha_1 \mid n \in \mathbb{N}\}$. Deduce that $G \simeq D_\infty$.

Exercise 2.3. Consider the Coxeter systems (W_1, S_1) with Coxeter graph Γ_1 and (W_2, S_2) with Coxeter graph Γ_2:

Γ_1: 

Γ_2: 

(a) Show that $W_1 \simeq \mathcal{D}_3 \times C_2$ and that $W_2 \simeq \mathcal{D}_3 \times C_2$ (consider the subgroup generated by $\sigma = (st)^3$).

(b) Conclude that the Coxeter group $\mathcal{D}_3 \times C_2$ admits two non-isomorphic Coxeter systems.

Exercise 2.4 (*)**.** Let V be the Euclidean plane. Let r be a linear rotation of angle θ and $G = \langle\, r \,\rangle$. Denote by $\mathscr{C}$ the unit circle.

(1) Assume that G is finite. Show that $\theta = \frac{2n\pi}{m}$, where the integers n and m are coprime and $m \ge 0$ and that $m = \mathrm{ord}(r)$.

(2) Show that any subgroup of $(\mathbb{R}, +)$ is either dense in $\mathbb{R}$ or of the form $a\mathbb{Z}$ for some $a > 0$.

(3) Assume that G is infinite.

(i) Show that $\theta = 2\pi/a$ with a irrational and that the group $\theta\mathbb{Z} + 2\pi\mathbb{Z}$ is dense into $\mathbb{R}$.

(ii) Prove that any G-orbit is dense in $\mathscr{C}$. (Hint: identify V with $\mathbb{C}$ and use the map $x \mapsto \exp(ix)$)

(4) Let r, s be two distinct linear reflections on V and $W = \langle\, s, r \,\rangle$ be the corresponding dihedral subgroup of the orthogonal group $\mathrm{O}(V)$. Show that W is infinite if and only if any W-orbit is dense in $\mathscr{C}$.

3. On the exchange condition

Let (W, S) be an involution system satisfying the *exchange condition*. We denote by Γ the Coxeter type of (W, S) with Coxeter matrix $(m_{st})_{s,t\in S}$.

We denote by $\ell : W \longrightarrow \mathbb{N}$ the length function on W. The *left descent set* (resp. the *right descent set*) of $w \in W$ is

$$D_L(w) = \{s \in S \mid \ell(sw) < \ell(w)\} \quad (\text{resp.} \quad D_R(w) = \{s \in S \mid \ell(ws) < \ell(w)\}).$$

Exercise 3.1. Show that (W, S) has a sign character.

Exercise 3.2. Show that if $w \in W$ and $s, r \in S$ such that $s \notin D_R(w)$ and $s \in D_R(rw)$ then $rw = ws$.

Exercise 3.3 (*)**.** (1) Let $w = uv$ be a reduced product. Let $s \in D_L(w) \setminus D_L(u)$, and let $v = s_1 \cdots s_p$ be a reduced word. Show that there exists $1 \leq i \leq p$ such that

$$w = su\, s_1 \cdots \widehat{s_i} \cdots s_p$$

is a reduced product.

(2) Let $w \in W$. Suppose that $s, t \in D_L(w)$ for $s, t \in S$ distinct. Show that there exists $u \in W$ such that $w = stu$ and $\ell(w) = \ell(u) + 2$. In other words, the product $w = stu$ is reduced.

(3) Let $w \in W$ and let $s, t \in D_L(w)$ distinct. Set

$$E_{st} = \{u \in W \mid w = [s, t]_k u \text{ is reduced}, \quad 2 \leq k \leq m_{st}\}.$$

(i) Show that $E_{st} \neq \varnothing$.

(ii) Let $u \in E_{st}$ be of minimal length. Show that $m_{st} \in \mathbb{N}_{\geq 2}$ and that

$$w = [s, t]_{m_{st}} u = [t, s]_{m_{st}} u$$

are reduced products.

(iii) Deduce that there exist $s_1, \ldots, s_q \in S$ such that

$$w = [s, t]_{m_{st}} s_1 \cdots s_q = [t, s]_{m_{st}} s_1 \cdots s_q$$

are two reduced words.

(4) Show that (W, S) satisfies the braid condition.

Hint. Proceed by induction, setting $W_k = \{w \in W \mid \ell(w) \leq k\}$.

4. On the braid condition with sign character

Let (W, S) be an involution system with a *sign character* and satisfying the *braid condition*. We denote by $\ell : W \longrightarrow \mathbb{N}$ the length function on W.

Exercise 4.1. Consider the monoid $(\mathcal{P}(S), \cup)$ and fix $I \subseteq S$.

(1) Show that there is a map $\psi : W \to \mathcal{P}(S)$ mapping $w = s_1 \cdots s_k$ (reduced word) to $\{s_1, \cdots, s_k\}$.

(2) Let $w \in W_I$ and consider a reduced word of $w = s_1 \cdots s_k$ of w in the involution system (W_I, I). Show this word is also reduced in (W, S), that is, $\ell_{(W_I,I)} = \ell_{(W,S)}$. (Hint: proceed by induction on k.)

(3) Deduce that if $w = s_1 \cdots s_k$ is a reduced word in (W, S), then $s_i \in I$ for any $1 \leq i \leq k$.

Exercise 4.2 (Matsumoto theorem)**.** Show that any reduced words for an element $w \in W$ are obtained from each other by a sequence of braid relations. (Hint. Consider the positive Artin-Tits monoid $\mathcal{B}_S^+$ and interpret classes of words up to braid relation in that monoid.)

5. On the characterization of Coxeter systems

Exercise 5.1. Let (W, S) be an involution system. Show that:

(1) If (W, S) satisfies the exchange condition, then (W, S) satisfies the braid relation and has a sign character.

(2) If (W, S) satisfies the braid relation and has a sign character, then (W, S) is a Coxeter system.

(Christophe Hohlweg) PK-4230
Email address: hohlweg.christophe@uqam.ca