EXAMEN DE MI-SESSION - MIDTERM EXAM,
MAT995Q–COXETER GROUPS
UQAM 2026

Ce devoir peut être rédigé soit en français ou soit en anglais. This homework can be written either in French or in English.

- Les exercices marqués d'un [∗] sont plus difficiles et doivent être choisis en priorité par les personnes étudiantes au doctorat. Exercises marked with a [∗] are more difficult and must be chosen first by students at the PhD level.
- Les exercices peuvent être résolus en groupe ou individuellement. Chaque personne étudiante choisit un exercice et en rédige *seule* la solution; la solution d'un même exercice ne peut pas être rédigée par deux personnes étudiantes distinctes. Chaque solution devra être rédigée sous LaTeXdans la class *AMSART* en trois pages maximum. La version PDF peut être remise au lieu d'une copie papier. **Le devoir doit être remis le 23 novembre.**
- The exercises might be solved individually or in groups. Each student will then choose one problem for which they will write the solution; the solution to the same problem cannot be written by two different students. The homework solution must be written in LaTeXusing the *AMSART* class in a maximum of three pages. A PDF version might be submitted instead of the paper version. **The homework has to be handed in by November 23rd.**

Notation and terminology. Let (W, S) be an involution system. We denote by $\ell : W \longrightarrow \mathbb{N}$ the length function on W. The *left descent set* (resp. the *right descent set*) of $w \in W$ is

$$D_L(w) = \{s \in S \mid \ell(sw) < \ell(w)\} \quad (\text{resp.} \quad D_R(w) = \{s \in S \mid \ell(ws) < \ell(w)\}).$$

We denote also $T = \bigcup_{w \in W} wSw^{-1}$ its set of reflections. The set of *inversion of* $w \in W$ is the set $T(w) = \{t \in T \mid \ell(tw) < \ell(w)\}$.

The group W acts on $\mathcal{P}(T)$ by conjugation: for $w \in W$ and $A \in \mathcal{P}(T)$, write:

$$w \cdot A = \{wtw^{-1} \mid t \in A\} \in \mathcal{P}(T).$$

The power set $\mathcal{P}(X)$ of a set X with the symmetric difference $+$ is an abelian group: for $A, B \in \mathcal{P}(X)$ we write $A + B = (A \setminus B) \sqcup (B \setminus A)$. For or $A_1, \dots A_n \in \mathcal{P}(X)$, we recall that:

$$A_1 + \cdots + A_n = \{x \in X \mid |\{1 \leq i \leq n \mid x \in A_i\}| \text{ is odd } \}.$$

1. On the exchange condition

Let (W, S) be an involution system satisfying the *exchange condition*. We denote by Γ the Coxeter type of (W, S) with Coxeter matrix $(m_{st})_{s,t \in S}$.

Exercise 1.1. (1) Show that (W, S) has a sign character.
(2) Show that if $w \in W$ and $s, r \in S$ such that $s \notin D_R(w)$ and $s \in D_R(rw)$ then $rw = ws$.
(3) Let $w = uv$ be a reduced product. Let $s \in D_L(w) \setminus D_L(u)$, and let $v = s_1 \cdots s_p$ be a reduced word. Show that there exists $1 \leq i \leq p$ such that

$$w = su\, s_1 \cdots \widehat{s_i} \cdots s_p$$

is a reduced product.

Exercise 1.2 (∗)**.** (1) Let $w \in W$. Suppose that $s, t \in D_L(w)$ for $s, t \in S$ distinct. Show that there exists $u \in W$ such that $w = stu$ and $\ell(w) = \ell(u) + 2$. In other words, the product $w = stu$ is reduced.

(2) Let $w \in W$ and let $s, t \in D_L(w)$ distinct. Set

$$E_{st} = \{u \in W \mid w = [s,t]_k u \text{ is reduced}, \quad 2 \leq k \leq m_{st}\}.$$

(i) Show that $E_{st} \neq \varnothing$.
(ii) Let $u \in E_{st}$ be of minimal length. Show that $m_{st} \in \mathbb{N}_{\geq 2}$ and that

$$w = [s,t]_{m_{st}} u = [t,s]_{m_{st}} u$$

are reduced products.
(iii) Deduce that there exist $s_1, \ldots, s_q \in S$ such that

$$w = [s,t]_{m_{st}} s_1 \cdots s_q = [t,s]_{m_{st}} s_1 \cdots s_q$$

are two reduced words.
(3) Show that (W,S) satisfies the braid condition.

Hint. Proceed by induction, setting $W_k = \{w \in W \mid \ell(w) \leq k\}$.

2. On the braid condition with sign character

Let (W,S) be an involution system with a *sign character* and satisfying the *braid condition*.

Exercise 2.1. Consider the monoid $(\mathcal{P}(S), \cup)$ and fix $I \subseteq S$.

(1) Show that there is a map $\psi : W \to \mathcal{P}(S)$ mapping $w = s_1 \cdots s_k$ (reduced word) to $\{s_1, \cdots, s_k\}$.
(2) Let $w \in W_I$ and consider a reduced word of $w = s_1 \cdots s_k$ of w in the involution system (W_I, I). Show this word is also reduced in (W,S), that is, $\ell_{(W_I,I)} = \ell_{(W,S)}$. (Hint: proceed by induction on k.)
(3) Deduce that if $w \in W_I$ and $w = s_1 \cdots s_k$ is a reduced word in (W,S), then $s_i \in I$ for every $1 \leq i \leq k$.

Exercise 2.2. Show that:

(1) any reduced words for an element $w \in W$ are obtained from each other by a sequence of braid relations. (Hint. Consider the positive Artin-Tits monoid $\mathcal{B}_S^+$ and interpret classes of words up to braid relation in that monoid.)
(2) (W,S) is a Coxeter system.

3. The reflection cocycle condition

We say that an involution system (W,S) has the *reflection cocycle condition* if there is a map $\mathcal{N} : W \to \mathcal{P}(T)$ such that $\mathcal{N}(s) = \{s\}$ for all $s \in S$ and $\mathcal{N}(uv) = \mathcal{N}(u) + u \cdot \mathcal{N}(v)$ for all $u, v \in W$.

Exercise 3.1 (*)**.** Let (W,S) be an involution system satisfying the reflection cocycle condition.

The aim of this exercise is to show that a reflection cocycle is uniquely determined by (W,S) *and that* $|\mathcal{N}(w)| = \ell(w)$ *for all* $w \in W$.

(a) Show that $\mathcal{N}(e) = \emptyset$ and that $\mathcal{N}(w) = w \cdot \mathcal{N}(w)$ for any involution $w \in W$.
(b) Let $n \in \mathbb{N}^*$ and $r_1, \ldots, r_n \in T$, show that

$$\mathcal{N}(r_1 r_2 \ldots r_n) = \mathcal{N}(r_1) + r_1 \cdot \mathcal{N}(r_2) + \cdots + r_1 \ldots r_{n-1} \cdot \mathcal{N}(r_n).$$

(c) Let $n \in \mathbb{N}^*$ and let $w = s_1 \ldots s_k \in W$ be a (non necessarily reduced) word. Define for $1 \leq i \leq k$ the following reflections: $t_1 = s_1,\ t_2 = s_1 s_2 s_1,\ \ldots,\ t_k = s_1 \ldots s_{k-1} s_k s_{k-1} \ldots s_1$. Show that:

$$\mathcal{N}(w) = \{t_1\} + \{t_2\} + \cdots + \{t_k\} = \{t \in T \mid |\{1 \leq i \leq n \mid t = t_i\}| \text{ is odd }\}.$$

(d) Show that $\mathcal{N}$ is the unique reflection cocycle of (W,S).
(e) Show that $\ell(w) = |\mathcal{N}(w)|$.

Exercise 3.2 (*)**.** Let (W, S) be a involution system and $\mathcal{N} : W \to \mathcal{P}(T)$ a reflection cocycle. We assume the results of Exercise 3.1 to be known for this exercise.

The aim of this exercise is to show that an involution system that has a reflection cocycle is a Coxeter system.

(a) Let $t \in T$, show that $t \in \mathcal{N}(t)$.
(b) Prove that $\mathcal{N}(w) = T(w)$ and $T \setminus T(w) = \{t \in T \mid \ell(tw) > \ell(w)\}$.
(c) Show that (W, S) satisfies the strong exchange condition, i.e, if $w = s_1 \cdots s_k$ is a reduced word for $w \in W$ and $t \in T(w)$ then there is $1 \leq i \leq k$ such that $w = ts_1 \cdots \hat{s}_i \cdots s_k$. Deduce that (W, S) is a Coxeter system.
(d) Let $t = s_1 \dots s_n$ be a reduced word for $t \in T$. Show there is $k \in \mathbb{N}$ such that $n = 2k + 1$, then prove that $t = s_1 \dots s_k s_{k+1} s_k \dots s_1$ is a palindromic reduced word for t.

4. Exercices on Coxeter systems

From now on, assume the results of the preceeding section to be known.

Exercise 4.1 (*)**.** Let (W, S) be a Coxeter system with its reflection cocycle.

The aim of this exercise is to show that a reflection subgroup in a Coxeter system admits itself a structure of Coxeter system.

For a subgroup H in W we set:

- $T_H(w) = T(w) \cap H$, for any $w \in W$;
- $S(H) = \{t \in T \mid T_H(t) = \{t\}\}$ and $T_H = H \cap T$.

A subgroup W' of W generated by some $U \subseteq T$ is a *reflection subgroup of* (W, S).

(a) Let H be a subgroup of W and $s \in S$. Show that either $S(sHs) = S(H)$ if $s \in H$ or $S(sHs) = sS(H)s$ if $s \notin H$.
(b) Show that for any subgroup H of W and $t \in T_H$, there is $t_0, \dots, t_m \in S(H)$ such that $t = t_m \dots t_1 t_0 t_1 \dots t_m$ (Proceed by induction on $\ell(t)$).
(c) Let $H \leq W$, $h \in H$ and $w \in W$. Show that $T_H(hw) = T_H(h) + h \cdot T_H(w)$.
(d) Let W' be a reflection subgroup of (W, S). Show that $(W', S(W'))$ is a Coxeter system with set of reflections $T_{W'}$ and reflection cocycle defined by $w \mapsto T_{W'}(w)$.

Exercise 4.2. Let (W, S) be a Coxeter system endowed with the right weak order $(W, \leq_R)$.

(a) For $u, v \in W$, prove that $d_W(u, v) = |T(u) + T(v)|$, where d_W is the word metric on $\mathrm{Cay}(W, S)$.

Let $w \in W$ and consider $I_w = [e, w]_R = \{x \in W \mid x \leq_R w\}$, an interval in the right weak order $(W, \leq_R)$.

(b) Show that $d_W(u, v) \leq \ell(w)$ for all $u, v \in I_w$.
(c) Let $u, v \in I_w$, show that $d_W(u, v) = \ell(w)$ if and only if $T(w) = T(u) \sqcup T(v)$.
(d) Assume that $|D_R(w)| = 1$ or $|D_L(w)| = 1$. Let $u, v \in I_w$ satisfy $d_W(u, v) = \ell(w)$. Show that $\{u, v\} = \{e, w\}$.

Exercise 4.3 (Longest element of a finite standard parabolic subgroup)**.** Let (W, S) be a Coxeter system, and let $I \subseteq S$ such that W_I is finite. Set $T_I = T \cap W_I$. Choose $w_{0,I} \in W_I$ of maximal length.

(a) Show that $D_L(w_{0,I}) = D_R(w_{0,I}) = I$.
(b) Show that, for every $u \in W_I$, $\ell(w_{0,I}u) = \ell(uw_{0,I}) = \ell(w_{0,I}) - \ell(u)$. (Hint: use the exchange condition and induction on $\ell(u)$.)
(c) Deduce that $w_{0,I}$ is the unique element of maximal length in W_I, that $w_{0,I}^2 = e$, and that

$$T(w_{0,I}) = T_I.$$

(d) Show that $W_I = [e, w_{0,I}]_R$. Thus $w_{0,I}$ is the greatest element of W_I for the right weak order.
(e) Let $w \in W$ and $K \subseteq D_L(w)$. Show that W_K is finite.

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